

New Fundamental Symmetries of Integrable Systems and Partial Bethe Ansatz¹

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Abstract

We introduce a new concept of quasi-Yang-Baxter algebras. The quantum quasi-Yang-Baxter algebras being simple but non-trivial deformations of ordinary algebras of monodromy matrices realize a new type of quantum dynamical symmetries and find an unexpected and remarkable applications in quantum inverse scattering method (QISM). We show that applying to quasi-Yang-Baxter algebras the standard procedure of QISM one obtains new wide classes of quantum models which, being integrable (i.e. having enough number of commuting integrals of motion) are only quasi-exactly solvable (i.e. admit an algebraic Bethe ansatz solution for arbitrarily large but limited parts of the spectrum). These quasi-exactly solvable models naturally arise as deformations of known exactly solvable ones. A general theory of such deformations is proposed. The correspondence “Yangian — quasi-Yangian” and “XXX spin models — quasi-XXX spin models” is discussed in detail. We also construct the classical counterparts of quasi-Yang-Baxter algebras and show that they naturally lead to new classes of classical integrable models. We conjecture that these models are quasi-exactly solvable in the sense of classical inverse scattering method, i.e. admit only partial construction of action-angle variables.

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1 Introduction

The quantum inverse scattering method (QISM) formulated in the works [31, 27, 8, 32, 21, 33] of the Leningrad group is at present time the biggest factory of quantum integrable and exactly solvable models most of which are of a great physical and mathematical relevance (see e.g. refs. [22, 20, 9, 10] and references therein). The main idea of this method rests on the use of a special associative algebra \mathcal{T}_R (known in the literature under names of Yang–Baxter algebra, Zamolodchikov algebra, Fundamental Commutation Relations, etc.)³, whose generators $T_{\alpha\beta}$, $\alpha, \beta = 1, \dots, d$ (forming the so-called *monodromy matrix*) satisfy the system of bilinear relations

$$R_{\alpha\beta\gamma\delta}(\lambda - \mu)T_{\gamma\rho}(\lambda)T_{\delta\sigma}(\mu) = T_{\beta\delta}(\mu)T_{\alpha\gamma}(\lambda)R_{\gamma\delta\rho\sigma}(\lambda - \mu). \quad (1.1)$$

Tensor $R_{\alpha\beta\gamma\delta}(\lambda)$ (which is usually called the R -matrix) obeys the famous Yang–Baxter equation and completely determines the structure of algebra \mathcal{T}_R . The role of this algebra in QISM can be explained as follows. By using relations (1.1) one easily constructs the families of operators

$$H(\lambda) = \mathcal{P}[\{T_{\alpha_i\beta_i}(\lambda_i)\}] \quad (1.2)$$

commuting with each other for any values of λ . Here $\mathcal{P}[\dots]$ denote some polynomials in generators $T_{\alpha_i\beta_i}(\lambda_i)$ and λ_i are some fixed functions of λ . These operators are considered as integrals of motion of some quantum systems. To solve an eigenvalue problem for these systems one uses a special ansatz

$$\psi = \mathcal{Q}[\{T_{\alpha_i\beta_i}(\xi_i)\}]|0\rangle \quad (1.3)$$

(the so-called Bethe ansatz) in which $|0\rangle$ is the so-called vacuum vector (playing the role of the lowest weight vector for the representation space of algebra \mathcal{T}_R) and $\mathcal{Q}[\dots]$ are again some polynomials in generators $T_{\alpha_i\beta_i}(\xi_i)$ taken at certain points ξ_i . The coordinates of these points are found from a system of numerical equations known under name of Bethe ansatz equations. Summarizing, one can say that both the construction and solution of quantum problems in QISM is a *purely algebraic procedure* whose realization requires the knowledge of only two things: the defining relations of algebra \mathcal{T}_R itself and the properties of its lowest weight representations. It turns out that most of the models obtained and solved in such a way are *integrable* (i.e. admit enough number of commuting integrals of motion) and *exactly solvable* (i.e. can be solved algebraically for the whole spectrum)⁴. This enables one to qualify the \mathcal{T}_R algebra as a generator of *integrable* and *exactly solvable* quantum problems.

In this paper we intend to introduce a new concept of “quasi- \mathcal{T}_R algebras”. These algebras are distinguished by the fact that “almost” the same construction which led in the case of the ordinary \mathcal{T}_R algebras to exactly solvable models, leads in the case of quasi- \mathcal{T}_R algebras to

³Algebra \mathcal{T}_R appeared first in Yang’s papers [50, 51] as algebra of non-relativistic S -matrices and in Baxter’s works [2, 3] as algebra of transfer matrices. Later it reappeared in the papers by Zamolodchikov and Zamolodchikov [52, 53] as algebra of relativistic S -matrices. The most regular way of introducing \mathcal{T}_R algebras was formulated by Faddeev’s group [31, 27, 8, 32, 21, 33, 28, 29] and also by Drinfeld [5, 6] and Jimbo [17] who proved that \mathcal{T}_R is a Hopf algebra.

⁴In quantum mechanics the notions of integrability and exact solvability do not coincide because of the absence of a quantum analogue of the Liouville theorem.

integrable but only *quasi-exactly solvable* models⁵, i.e. models admitting an algebraic solution only for some limited parts of the spectrum⁶. The quasi- \mathcal{T}_R algebra is a very simple but non-trivial deformation of the ordinary \mathcal{T}_R algebra. The form of the commutation relations for \mathcal{T}_R and quasi- \mathcal{T}_R algebras “almost” coincide and all their properties (including construction of both commuting integrals of motion and their solutions) are “almost” the same. However, namely this “slight” difference makes the quasi- \mathcal{T}_R algebra a new mathematical object having many new exciting mathematical properties and a wide range of possible physical applications. The simplest example of quasi- \mathcal{T}_R algebra has already been considered in our previous paper [48].

Quasi- \mathcal{T}_R algebra is not an absolutely unexpected thing. The feeling that something like this may exist immediately appears if one remembers some facts from the theory of exactly and quasi-exactly solvable problems. Indeed, consider for example a pair of two simple quantum-mechanical models with hamiltonians

$$H = -\sum_{i=1}^d \frac{\partial^2}{\partial x_i^2} + \sum_{i=1}^d b_i^2 x_i^2 \quad (1.4)$$

and

$$H_n = -\sum_{i=1}^d \frac{\partial^2}{\partial x_i^2} + \sum_{i=1}^d b_i^2 x_i^2 + 2ar^2 \left[\sum_{i=1}^d b_i x_i^2 - 2n - 1 - \frac{d}{2} \right] + a^2 r^6, \quad n = 0, 1, \dots \quad (1.5)$$

The first model is known to everybody. This is an ordinary d -dimensional harmonic oscillator which is *integrable* (since it admits a complete separation of variables) and *exactly solvable* (since all its eigenvalues and eigenfunctions can be constructed algebraically). The second model of a d -dimensional anharmonic oscillator, which appeared on the physical scene only recently [41] (see also [43, 46]), is also *integrable* (because its variables can be separated in the generalized ellipsoidal coordinates [42]) but, in contrast with model (1.4), it is only *quasi-exactly solvable* (since admits only partial algebraic solution of spectral problem for any non-negative integer n)⁷.

What can we learn from comparing these two models? First of all, we see that model (1.5) can be considered as a *deformation* of the model (1.4) and the role of the deformation parameter is played by a . Second, it is clear that this deformation preserves the integrability property. Third, this deformation is of the splitting type, because it transforms a single exactly solvable model into an infinite sequence of quasi-exactly solvable models with different hamiltonians and different number of exact solutions [46].

It turns out that this situation is quite general. In fact, it can be shown that any integrable and exactly solvable model satisfying some simple and rather general conditions can be constructively deformed into an infinite sequence of integrable and quasi-exactly solvable models [46]. In particular, this is true for models associated with various \mathcal{T}_R algebras and

⁵The notion of “quasi-exact solvability” was introduced in paper [36].

⁶Numerous examples of quasi-exactly solvable models and various methods for their construction can be found in several review articles [43, 25, 45, 39, 26, 38, 15] and in the book [46].

⁷Indeed, it can be explicitly shown that, for any given n , the number of algebraically constructable solutions of model (1.5) is only $(n+d)!/(n!d!)$.

obtainable by means of QISM⁸. But if so, may be this deformation can be understood on a purely *algebraic* level? May be, there exists some non-trivial deformation of the underlying \mathcal{T}_R algebra, which could be called “quasi- \mathcal{T}_R algebra”, and which would naturally lead to quasi-exactly solvable versions of integrable models?

The aim of this paper is to give a positive answer to this question. To make the text more readable, we devote the main attention to the simplest rational \mathcal{T}_R -algebra corresponding to a monodromy matrix of the size 2×2 . The rationality means that both generators $T_{\alpha\beta}(\lambda)$, $\alpha, \beta = 1, 2$ and tensor $R_{\alpha\beta\gamma\delta}(\lambda)$, $\alpha, \beta, \gamma, \delta = 1, 2$ are rational functions of λ . Such an algebra is called the Yangian $\mathcal{Y}[sl(2)]$ and is characterized by the following choice of non-zero elements of tensor $R_{\alpha\beta\gamma\delta}(\lambda)$

$$\begin{aligned} R_{1111}(\lambda) = R_{2222}(\lambda) &= 1, \\ R_{1212}(\lambda) = R_{2121}(\lambda) &= \lambda/(\lambda + \eta), \\ R_{1221}(\lambda) = R_{2112}(\lambda) &= \eta/(\lambda + \eta), \end{aligned} \tag{1.6}$$

where η is a parameter. For $\eta \neq 0$ the Yangian $\mathcal{Y}[sl(2)]$ is a non-commutative and non-cocommutative Hopf algebra [5, 6]. In the limit $\eta \rightarrow 0$ it degenerates into a special (infinite-dimensional) Lie – Gaudin algebra [13] which we denote by $\mathcal{G}[sl(2)]$. The completely integrable and exactly solvable models associated with algebras $\mathcal{G}[sl(2)]$ and $\mathcal{Y}[sl(2)]$ are respectively known under names of Gaudin and XXX spin models.

The paper is organized as follows. After short exposition of properties of ordinary algebras $\mathcal{G}[sl(2)]$ and $\mathcal{Y}[sl(2)]$ (sections 2 and 8) we explain the reader what we mean under quasi- $\mathcal{G}[sl(2)]$ and quasi- $\mathcal{Y}[sl(2)]$ algebras and present their defining relations (sections 3 and 9). We also give a proof of the existence of these algebras by constructing their explicit realizations (sections 5 and 11). In particular, we demonstrate that they actually can be considered as deformations of algebras $\mathcal{G}[sl(2)]$ and $\mathcal{Y}[sl(2)]$ and show that these deformations are of the splitting type, i.e. the generators of “quasi” algebras contain an additional subscript n . The non-triviality lies however in the way in which this subscript appears in the defining relations (1.1) which, as for the rest, are the same as in the undeformed case. We show also that the proposed deformation preserves the integrability property (sections 3 and 9). Following general prescriptions of QISM, we construct the infinite series of the associated integrable and quasi-exactly solvable models (sections 3,6 and 9,12) and present their partial solutions in the framework of the algebraic Bethe ansatz (sections 4 and 10). We call the models obtained in such a way the quasi-Gaudin and quasi- XXX models, respectively.

In sections 13 – 16 we discuss the obtained results from the point of view of the existing methods in the theory of quasi-exactly solvable systems. We consider three general methods known under names of “partial algebraization method” (section 14), “inverse method of separation of variables” (section 15) and “projection method” (section 16), and show that, in contrast with the first and second methods, which are either not applicable or only partially applicable to the study of quasi- \mathcal{T}_R algebras, the third one enables one to reproduce correctly practically all features of the proposed formalism. However, the mathematical technique used in this method is so cumbersome and model-dependent that it hardly can be practically used beyond the simplest cases of algebras $\mathcal{G}[sl(2)]$ and $\mathcal{Y}[sl(2)]$ and their “quasi” analogues.

⁸The case of the simplest models of such a sort (the so-called Gaudin models [13]) was considered in refs. [45, 46]. For more general discussion of this subject see sections 15 and 16 of present paper.

Fortunately, the “quasi-QISM” formalism which we propose in this paper is free of all these deficiencies. Indeed, in sections 17 – 21 we show that this formalism is quite general and applicable not only to $\mathcal{G}[sl(2)]$ and $\mathcal{Y}[sl(2)]$ algebras, but also to all \mathcal{T}_R algebras, irrespective of their dimension and a concrete form of tensor $R_{\alpha\beta\gamma\delta}(\lambda)$. For this we will use a model independent formalism based on the notion of Z^r -graded unital associative algebras (which in this paper we call simply A -algebras) and construct quasi-deformations immediately for the latter. The obtained results are automatically applicable to any \mathcal{T}_R algebras since they can be considered as particular cases of A -algebras. This gives us the hope that using this general deformation technique we will find a lot of new quasi-exactly solvable models interesting both from physical and mathematical points of view.

One of questions discussed in this paper (see sections 7 and 20) is related to a special limit of quasi- \mathcal{T}_R algebras when the so-called “spectral parameter” λ characterizing their generators tends to infinity. We know that in the case of ordinary (classical or quantum) \mathcal{T}_R algebras such a limit leads respectively to ordinary finite-dimensional Lie algebras or their q -deformed (Lie_q) versions. It turns out that performing the same limiting procedure with quasi- \mathcal{T}_R algebras we also obtain some new interesting mathematical structures which can be called quasi-Lie and quasi-Lie $_q$ algebras.

In section 22 we introduce the concept of classical quasi- \mathcal{T}_R algebras. These algebras naturally arise as the result of *dequantization* of quantum quasi- \mathcal{T}_R algebras discussed in preceding sections. Their generators are realized as functions on the phase space and obey the special classical commutation relations. The latter are expressed through a special non-standard (quasi-Poisson) bracket which can be considered as a deformation of the ordinary Poisson bracket. We show that the role of classical quasi- \mathcal{T}_R algebras in the classical inverse scattering method (CISM) is the same as that of quantum quasi- \mathcal{T}_R algebras in QISM. The classical models associated with classical quasi- \mathcal{T}_R algebras are integrable but it seems that they are only partially solvable in the framework of CISM.

In the concluding section 23 we give a list of problems which naturally arise from the results of the present paper and, in our opinion, deserve a very careful study.

2 Gaudin algebra and Gaudin model

In this section we remind the reader some basic facts concerning Gaudin algebras, their representations and properties of the associated Gaudin models. More detailed exposition of this subject can be found in refs. [13, 19, 45, 46, 12].

As we mentioned in Introduction, the Gaudin algebra $\mathcal{G}[sl(2)]$ can be obtained from the Yangian $\mathcal{Y}[sl(2)]$ (which is characterized by R -matrix of the form (1.6) and commutation relations (1.1)) in the limit when $\eta \rightarrow 0$. For this the generators $T_{\alpha\beta}(\lambda)$, $\alpha, \beta = 1, 2$ should be taken in the form

$$\begin{aligned} T_{11}(\lambda) &= 1 + \eta S^0(\lambda) + \mathcal{O}(\eta^2), \\ T_{22}(\lambda) &= 1 - \eta S^0(\lambda) + \mathcal{O}(\eta^2), \\ T_{12}(\lambda) &= -\eta S^-(\lambda) + \mathcal{O}(\eta^2), \\ T_{21}(\lambda) &= +\eta S^+(\lambda) + \mathcal{O}(\eta^2), \end{aligned} \tag{2.1}$$

where $S^0(\lambda)$, $S^-(\lambda)$ and $S^+(\lambda)$ are just the generators of $\mathcal{G}[sl(2)]$. These generators, which are respectively called neutral, lowering and raising generators, obey the following commutation relations

$$\begin{aligned}
S^0(\lambda)S^0(\mu) - S^0(\mu)S^0(\lambda) &= 0, \\
S^+(\lambda)S^+(\mu) - S^+(\mu)S^+(\lambda) &= 0, \\
S^-(\lambda)S^-(\mu) - S^-(\mu)S^-(\lambda) &= 0, \\
S^0(\lambda)S^+(\mu) - S^+(\mu)S^0(\lambda) &= -(\lambda - \mu)^{-1}\{S^+(\lambda) - S^+(\mu)\}, \\
S^0(\lambda)S^-(\mu) - S^-(\mu)S^0(\lambda) &= +(\lambda - \mu)^{-1}\{S^-(\lambda) - S^-(\mu)\}, \\
S^-(\lambda)S^+(\mu) - S^+(\mu)S^-(\lambda) &= -2(\lambda - \mu)^{-1}\{S^0(\lambda) - S^0(\mu)\}.
\end{aligned} \tag{2.2}$$

From (2.2) it immediately follows that operators

$$H(\lambda) = S^0(\lambda)S^0(\lambda) - \frac{1}{2}S^-(\lambda)S^+(\lambda) - \frac{1}{2}S^+(\lambda)S^-(\lambda) \tag{2.3}$$

form a commutative family,

$$[H(\lambda), H(\mu)] = 0, \tag{2.4}$$

and thus can be interpreted as integrals of motion of a certain quantum completely integrable model⁹. The latter is known under name of Gaudin model.

The role of the carrier space in which the operators $H(\lambda)$ act¹⁰ is played by the representation space of Gaudin algebra. In order to construct it one needs to fix the lowest weight vector $|0\rangle$ and the lowest weight function $F(\lambda)$ obeying the relations

$$S^0(\lambda)|0\rangle = F(\lambda)|0\rangle, \quad S^-(\lambda)|0\rangle = 0. \tag{2.5}$$

After this, we can define the representation space as a linear hull of vectors

$$|\xi_1, \dots, \xi_n\rangle = S^+(\xi_n) \dots S^+(\xi_1)|0\rangle, \tag{2.6}$$

with arbitrary n and ξ_1, \dots, ξ_n . We shall denote this space by $W_{F(\lambda)}$.

The ‘‘Schrödinger equation’’ for the Gaudin model reads now

$$H(\lambda)\phi = E(\lambda)\phi, \quad \phi \in W_{F(\lambda)}. \tag{2.7}$$

The beauty of this equation lies in the fact that all its solutions can be obtained algebraically within the so-called Bethe ansatz

$$\phi = S^+(\xi_n)S^+(\xi_{n-1}) \dots S^+(\xi_2)S^+(\xi_1)|0\rangle, \tag{2.8}$$

⁹Using concrete realizations of generators of algebra $\mathcal{G}[sl(2)]$, one can easily check that the number of independent operators belonging to the class $H(\lambda)$ is sufficiently large for claiming that this model is completely integrable.

¹⁰We try to avoid the use of the term ‘‘Hilbert space’’ because we do not intend to discuss in this paper the hermiticity properties of operators $H(\lambda)$ and the normalizability of their eigenstates.

in which n is an arbitrarily fixed non-negative integer and ξ_1, \dots, ξ_n are some still unknown numbers. To demonstrate this, one needs to act on Bethe vector by operator $H(\lambda)$ and, using commutation relations

$$H(\lambda)S^+(\xi_i) = S^+(\xi_i)H(\lambda) + 2\frac{S^+(\xi_i)S^0(\lambda) - S^+(\lambda)S^0(\xi_i)}{\lambda - \xi_i} \quad (2.9)$$

which elementary follow from (2.2), transfer $H(\lambda)$ to the right. The neutral operators $S^0(\lambda)$ and $S^0(\xi_i)$ appearing in the right hand side of (2.9) also should be transferred to the right. For this the fourth relation in the list (2.2) can be used. Finally, we obtain a number of terms with operators $H(\lambda)$, $S^0(\lambda)$ and $S^0(\xi_i)$, $i = 1, \dots, n$ acting immediately on the lowest weight vector $|0\rangle$. After this one should get rid of these operators by using formulas (2.5) together with the relation

$$H(\lambda)|0\rangle = (F^2(\lambda) + F'(\lambda))|0\rangle \quad (2.10)$$

which elementary follows from (2.5) and (2.3)¹¹. The result will consist of two parts. The first part will be proportional to the Bethe vector ϕ_n , while the second part will contain n terms not having the initial Bethe form (these are the so-called “unwanted terms”). The condition of cancellation of these terms imposes some special conditions on numbers ξ_i , $i = 1, \dots, n$ which up to now were considered as free parameters. An explicit form of these conditions, which are commonly known as Bethe ansatz equations, reads

$$\sum_{k=1, k \neq i}^n \frac{1}{\xi_i - \xi_k} + F(\xi_i) = 0, \quad i = 1, \dots, n. \quad (2.11)$$

Putting all these facts together, one can easily obtain the final solution of the Gaudin spectral problem:

$$E(\lambda) = F^2(\lambda) + F'(\lambda) + 2 \sum_{i=1}^n \frac{F(\lambda) - F(\xi_i)}{\lambda - \xi_i}. \quad (2.12)$$

It is obvious that if $F(\lambda)$ is a rational function, then the number of solutions of Bethe ansatz equations (2.11) is finite for any n . But since n is an arbitrary non-negative integer, the complete number of solutions of problem (2.7) is infinite. It can be shown that these solutions expire all possible solutions of problem (2.7) and therefore it can be qualified as exactly solvable.

3 Quasi-Gaudin algebra and quasi-Gaudin models

Let us now try to explain the reader what we mean under words “quasi-Gaudin algebra”. Note that this will be namely an attempt of an explanation and not a list of rigorous definitions and theorems. A strict mathematical interpretation of formulas of this section will be given later in sections 17 – 21.

¹¹In order to derive (2.10) one should first take the last relation in the list (2.2) in the limit when $\mu \rightarrow \lambda$, and using it rewrite the operator (2.3) in the form $H(\lambda) = [S^0(\lambda)]^2 + [S^0(\lambda)]' - S^+(\lambda)S^-(\lambda)$. After this one should act by this operator on the vacuum vector and then use formulas (2.5).

Let $S_n^0(\lambda)$, $S_n^-(\lambda)$ and $S_n^+(\lambda)$ denote the operators parametrized by an integer number n and obeying the relations

$$\begin{aligned}
S_n^0(\lambda)S_n^0(\mu) - S_n^0(\mu)S_n^0(\lambda) &= 0, \\
S_{n+1}^+(\lambda)S_n^+(\mu) - S_{n+1}^+(\mu)S_n^+(\lambda) &= 0, \\
S_{n-1}^-(\lambda)S_n^-(\mu) - S_{n-1}^-(\mu)S_n^-(\lambda) &= 0, \\
S_{n+1}^0(\lambda)S_n^+(\mu) - S_n^+(\mu)S_{n+1}^0(\lambda) &= -(\lambda - \mu)^{-1}\{S_n^+(\lambda) - S_n^+(\mu)\}, \\
S_{n-1}^0(\lambda)S_n^-(\mu) - S_n^-(\mu)S_{n-1}^0(\lambda) &= +(\lambda - \mu)^{-1}\{S_n^-(\lambda) - S_n^-(\mu)\}, \\
S_{n+1}^-(\lambda)S_n^+(\mu) - S_{n+1}^-(\mu)S_n^+(\lambda) &= -2(\lambda - \mu)^{-1}\{S_n^0(\lambda) - S_n^0(\mu)\}.
\end{aligned} \tag{3.1}$$

We consider (3.1) as the defining relations for a certain infinite-dimensional algebra which is very similar to the Gaudin algebra but, in contrast with the latter, it *is not* a Lie algebra. We shall call it “quasi- $\mathcal{G}[sl(2)]$ algebra” and the expressions staying in the left hand sides of relations (3.1) we refer to as “quasi-commutators”. A most unusual feature of algebra $\mathcal{G}[sl(2)]$ is that it is *partially free* i.e. does not contain quasi-commutation relations between generators whose indices differ more than by one. One can also say that this algebra is *local* in the space of a discrete parameter n . Note that the form of quasi-commutation relations is invariant under special “quasi-similarity” transformations

$$\begin{aligned}
S_n^-(\lambda) &\rightarrow U_{n-1}S_n^-(\lambda)U_{n-1}^{-1}, \\
S_n^0(\lambda) &\rightarrow U_nS_n^0(\lambda)U_n^{-1}, \\
S_n^+(\lambda) &\rightarrow U_{n+1}S_n^+(\lambda)U_{n+1}^{-1},
\end{aligned} \tag{3.2}$$

in which U_n are arbitrary invertible operators.

One of the remarkable properties of quasi- $\mathcal{G}[sl(2)]$ algebra is that its generators (exactly as in the $\mathcal{G}[sl(2)]$ algebraic case) can be used for building hamiltonians of some completely integrable quantum systems. Indeed, using quasi-commutation relations (3.1) it is not difficult to prove that the operators

$$H_n(\lambda) = S_n^0(\lambda)S_n^0(\lambda) - \frac{1}{2}S_{n+1}^-(\lambda)S_n^+(\lambda) - \frac{1}{2}S_{n-1}^+(\lambda)S_n^-(\lambda) \tag{3.3}$$

form commutative families for any n :

$$[H_n(\lambda), H_n(\mu)] = 0. \tag{3.4}$$

At the same time, the operators $H_n(\lambda)$ and $H_m(\mu)$ with different n and m do not generally commute with each other because of the absence of appropriate (non-local) relations in algebra $\mathcal{G}[sl(2)]$. This means that formula (3.3) describes in fact an infinite set of *different* integrable models¹². This is, may be, the most important difference with the Gaudin case where an analogous construction leads to a single completely integrable model — the Gaudin model.

Exactly as in the case of Gaudin algebra, the final construction of models with hamiltonians (3.3) needs the specification of a carrier space in which these hamiltonians act. It

¹²Later, in section 5, after constructing a concrete realization of algebra $\mathcal{G}[sl(2)]$ we will be able to demonstrate that operators $H_n(\lambda)$ and $H_m(\mu)$ actually do not commute with each other. At the same time, the number of independent operators contained in the family $H_n(\lambda)$ with a given n is sufficiently large to claim that the corresponding models are completely integrable.

would be natural to identify it with the “representation space” of our algebra¹³. The latter can be constructed almost in the same way as in the Gaudin case. There is however an essential difference which lies in the fact that the quasi-analogues of relations (2.5) cannot be written simultaneously for all “neutral” and “lowering” operators $S_n^0(\lambda)$ and $S_n^-(\lambda)$, i.e. for any n . This simply would lead to an overdetermined system of equations for the vacuum vector $|0\rangle$. Roughly speaking, to have the same number of equations for $|0\rangle$ as in (2.5), one should restrict ourselves to one arbitrarily fixed value of n . Taking for example $n = 0$, we obtain

$$S_0^0(\lambda)|0\rangle = F(\lambda)|0\rangle, \quad S_0^-(\lambda)|0\rangle = 0, \quad (3.5)$$

where $F(\lambda)$ is, as before, a certain function playing the role of the lowest weight. The impossibility of writing the relations like (3.5) for any n is another unusual property of quasi- $\mathcal{G}[sl(2)]$ algebra, and it, strictly speaking, should be postulated from very beginning¹⁴. The presence of such a postulate will play a very important role in our further considerations.

After fixing the “lowest weight vector” $|0\rangle$ together with the corresponding “lowest weight function” $F(\lambda)$, we can define the “representation space” by analogy with (2.6) i.e. as a linear hull of vectors obtained after action of operators $S_n^+(\xi_i)$ on $|0\rangle$. It is however intuitively clear that there is a big freedom in choosing such vectors because the indices of raising operators may take infinitely many different values. In order to restrict this freedom (which, unfortunately, we cannot control at present time), we need a “guiding principle”, which, in particular, could be based on a selectional use of some symmetries. Looking for example at the vectors (2.6) forming the representation space of ordinary Gaudin algebra, we see that, due to the commutativity of raising generators $S^+(\xi_1), \dots, S^+(\xi_n)$, they are symmetric under all permutations of numbers ξ_1, \dots, ξ_n . It is natural to demand the same symmetry in the case of quasi-Gaudin algebras. Now, however, the raising operators do not commute anymore, they only “quasi-commute”, and therefore the only way of having such a symmetry is to choose the corresponding vectors in the form

$$|\xi_1, \dots, \xi_m\rangle_k = S_{m+k}^+(\xi_m) S_{m-1+k}^+(\xi_{m-1}) \dots S_{2+k}^+(\xi_2) S_{1+k}^+(\xi_1) |0\rangle \quad (3.6)$$

with arbitrary $k \in \mathbb{Z}$, $m \in \mathbb{N}$ and ξ_1, \dots, ξ_m . As before, we denote the “representation space” of algebra (3.1) defined in such a way by $W_{F(\lambda)}$.

Now we can complete the construction of integrable models (3.3) associated with algebra (3.1). We postulate that “Schrödinger equations” for these models read

$$H_n(\lambda)\phi = E(\lambda)\phi, \quad \phi \in W_{F(\lambda)}, \quad n = 0, 1, 2, \dots \quad (3.7)$$

Hereafter we shall call such models the “quasi-Gaudin models”.

¹³Generally speaking, the object which we intend to call the “representation space” is not a true representation space for quasi- $\mathcal{G}[sl(2)]$ algebra since, even after choosing an appropriate basis in it, we cannot represent any element of this algebra in a matrix form. This is possible only for some special realizations of quasi- $\mathcal{G}[sl(2)]$, one of which will be considered in section 5.

¹⁴Later, in section 5 we demonstrate that the lowest weight vector $|0\rangle$ is actually not annihilated by the “lowering operators” $S_n^-(\lambda)$ if $n \neq 0$. Here, however, we suggest the reader to consider this fact as a postulate.

4 Bethe ansatz for quasi-Gaudin models

It turns out that, in full analogy with the Gaudin case, the solutions of spectral equations (3.7) can be found by means of the algebraic Bethe ansatz. But before constructing the appropriate Bethe vectors one should discuss some auxiliary problems concerning the properties of quasi-Gaudin algebra.

It is known that in the Gaudin case the lowest weight vector is always a solution of the Gaudin spectral problem (see for example formula (2.10)). Is this true in the case of quasi-Gaudin models? In order to answer this question, one should simply act by the operator $H_n(\lambda)$ on the vector $|0\rangle$ and look at the result. For this it is convenient to rewrite the operator $H_n(\lambda)$ in a little bit different form. Using the last formula in the list (3.1) and taking in it the limit $\mu \rightarrow \lambda$ we obtain

$$S_{n+1}^-(\lambda)S_n^+(\lambda) = S_{n-1}^+(\lambda)S_n^-(\lambda) - 2\frac{\partial}{\partial\lambda}S_n^0(\lambda). \quad (4.1)$$

Substituting (4.1) into expression (3.3) for $H_n(\lambda)$, we get the needed expression for $H_n(\lambda)$

$$H_n(\lambda) = S_n^0(\lambda)S_n^0(\lambda) + \frac{\partial}{\partial\lambda}S_n^0(\lambda) - S_{n-1}^+(\lambda)S_n^-(\lambda). \quad (4.2)$$

Remembering now that for $n \neq 0$ the vacuum vector $|0\rangle$ is *not* in general annihilated by the lowering operators $S_n^-(\lambda)$ and is *not* an eigenvector of neutral operators $S_n^0(\lambda)$, we can conclude that it may be an eigenvector of (4.2) only if $n = 0$. Using formulas (3.5), we obtain for this case the relation

$$H_0(\lambda)|0\rangle = \left(F^2(\lambda) + F'(\lambda)\right)|0\rangle \quad (4.3)$$

which is similar to the Gaudin relation (2.10).

Let us now return to the main problem of solving equations (3.7) by means of Bethe ansatz. It is natural to take the Bethe vector in the form

$$\phi = S_{m+k}^+(\xi_m)S_{m-1+k}^+(\xi_{m-1}) \dots S_{2+k}^+(\xi_2)S_{k+1}^+(\xi_1)|0\rangle \quad (4.4)$$

with some $k \in \mathbb{Z}$ and $m \in \mathbb{N}$, and, using the relations

$$H_n(\lambda)S_{n-1}^+(\xi_n) = S_{n-1}^+(\xi_n)H_{n-1}(\lambda) + 2\frac{S_{n-1}^+(\xi_n)S_{n-1}^0(\lambda) - S_{n-1}^+(\lambda)S_{n-1}^0(\xi_n)}{\lambda - \xi_n}, \quad (4.5)$$

which follow from the basic relations (3.1), try to transfer the operator $H_n(\lambda)$ to the right (exactly as in the ordinary Gaudin case). From formula (4.5) it is seen that, in order to start performing this procedure, it is necessary to take

$$m + k = n - 1. \quad (4.6)$$

Otherwise we would be unable to transfer the operator $H_n(\lambda)$ to the right because of the absence of the corresponding commutation relations in (4.5). From formula (4.5) it is readily seen that permutation of the operator $H_n(\lambda)$ with the first raising generator $S_{n-1}^+(\xi_n)$ of Bethe vector replaces $H_n(\lambda)$ by $H_{n-1}(\lambda)$. After this, $H_{n-1}(\lambda)$ will appear in front of the next

raising generator $S_{n-2}^+(\xi_{n-1})$, and we can again permute these operators by means of the same formula (4.5) but with n replaced by $n-1$. Obviously, this procedure can be continued further. The same relates to the neutral generators $S_{n-1}^0(\lambda)$ and $S_{n-1}^0(\xi_n)$ appearing in the right hand side of (4.5) and thus acting immediately on $S_{n-2}^+(\xi_{n-1})$. The fourth relation in the list (3.1) enables us to perform the permutation of these operators, after which the neutral generators appear in front of the generator $S_{n-3}^+(\xi_{n-2})$ and will have the index $n-2$. This enables one to use again the forth formula in (3.1), etc. Summarising, one can say that any permutation of H - and S^0 -operators with raising generators forming the Bethe vector decreases their indices by one. Finally, when all the transferences will be completed, we obtain a number of terms with operators $H_{n-m}(\lambda)$, $S_{n-m}^0(\lambda)$ and $S_{n-m}^0(\xi_i)$, $i = 1, \dots, n$ acting immediately on the lowest weight vector $|0\rangle$. The standard prescriptions to Bethe ansatz technique (see e.g. section 2) imply that all these operators should be absorbed by this vector. But such an absorbtion is possible only if the lowest weight vector is an eigenvector of these operators, which, according to the previous results, is possible only if

$$n - m = 0. \quad (4.7)$$

Comparing formulas (4.6) and (4.7) we can conclude that the only case when the ansatz (4.4) for equation (3.7) may lead to some algebraic solutions corresponds to the choice

$$k = -1, \quad m = n. \quad (4.8)$$

After using the restrictions (4.8) the ansatz (4.4) takes the following final form

$$\phi = S_{n-1}^+(\xi_n) S_{n-2}^+(\xi_{n-1}) \dots S_1^+(\xi_2) S_0^+(\xi_1) |0\rangle. \quad (4.9)$$

Now it remains only to check that it actually contains solutions of the quasi-Gaudin spectral problem (3.7). This can be demonstrated exactly in the same way as in the ordinary Gaudin case. Getting rid of the operators $H_0(\lambda)$, $S_0^0(\lambda)$ and $S_0^0(\xi_i)$, $i = 1, \dots, n$ standing in front of the vacuum vector by means of formulas (3.5) and (4.3), we obtain two sorts of terms. One of these terms will be proportional to the Bethe vector ϕ , while the other ones will represent the good old “unwanted terms”. It is not difficult to show that the condition of cancellation of the latter reads

$$\sum_{k=1, k \neq i}^n \frac{1}{\xi_i - \xi_k} + F(\xi_i) = 0, \quad i = 1, \dots, n, \quad (4.10)$$

and the final expression for the eigenvalues $E(\lambda)$ is given by

$$E(\lambda) = F^2(\lambda) + F'(\lambda) + 2 \sum_{i=1}^n \frac{F(\lambda) - F(\xi_i)}{\lambda - \xi_i}. \quad (4.11)$$

This gives us the desired algebraic solution of problem (3.7). From the above consideration it is clearly seen that models with hamiltonians $H_n(\lambda)$ are typical quasi-exactly solvable ones. Indeed, the linear hull of all admissible Bethe vectors (4.9) forms only a certain small subspace in the space of vectors of the general form (4.4). The number of explicit solutions of problems (3.7) is finite for any given n if $F(\lambda)$ is a rational function. At the same time the dimension of the space $W_{F(\lambda)}$ is generally infinite-dimensional. It is interesting that we obtained exactly the same set of solutions as in the case of ordinary Gaudin model. But now these solutions are distributed between different models.

5 The existence of quasi-Gaudin algebras

Everything what we said in sections 3 and 4 can be checked by means of direct calculations. The only thing which may seem dubious is a very existence of an algebra with such strange properties. In order to make sure that it does actually exist, one should construct at least one of its explicit realizations. One of possible realization can be constructed immediately from the generators of Gaudin algebra. Let us take the generators $S^0(\lambda), S^-(\lambda), S^+(\lambda)$ introduced in section 2 and assume that they act in the representation space characterized by the lowest weight function $F^0(\lambda)$. Let us also introduce a special limiting operator

$$S^0 = \lim_{\lambda \rightarrow \infty} \lambda S^0(\lambda). \quad (5.1)$$

Then the desired realization reads

$$\begin{aligned} S_n^-(\lambda) &= S^-(\lambda) + \frac{F^0 - S^0 + n}{\lambda - a}, \\ S_n^0(\lambda) &= S^0(\lambda) + \frac{F^0 - S^0 + n + b}{\lambda - a}, \\ S_n^-(\lambda) &= S^-(\lambda) + \frac{F^0 - S^0 + n + 2b}{\lambda - a}, \end{aligned} \quad (5.2)$$

where a and b are arbitrary complex parameters and $F^0 = \lim_{\lambda \rightarrow \infty} \lambda F^0(\lambda)$. It can be easily checked that operators (5.2) actually satisfy the quasi-commutation relations (3.1).

The representation space of this algebra coincides by construction with the representation space of the Gaudin algebra. The lowest weight vector $|0\rangle$ is the same as in section 3 which gives us the possibility of checking formulas (3.5) by using their Gaudin analogues (2.5). Taking into account that

$$S^0|0\rangle = F^0|0\rangle \quad (5.3)$$

and using formulas (2.5), we obtain:

$$S_n^0(\lambda)|0\rangle = \left(F^0(\lambda) + \frac{b+n}{\lambda-a}\right)|0\rangle, \quad S_n^-(\lambda)|0\rangle = \left(\frac{n}{\lambda-a}\right)|0\rangle. \quad (5.4)$$

We see that the lowest weight vector is actually not annihilated by “lowering” operators if $n \neq 0$. Taking $n = 0$ in (5.4) we obtain the expression for the lowest weight of quasi-Gaudin algebra

$$F(\lambda) = F^0(\lambda) + \frac{b}{\lambda - a}. \quad (5.5)$$

Now it is absolutely clear that algebra (3.1) is a continuous *deformation* of the Gaudin algebra. The role of the deformation parameter is played by a . If $a \rightarrow \infty$, then the n -dependence of the operators (5.2) disappears and they transform into the ordinary generators of Gaudin algebra. Respectively, the quasi-commutators in (3.1) become the ordinary ones. As to the formulas (3.5) defining the representations of algebra (3.1), they, in the limit $a \rightarrow \infty$, also transform into relations (2.5) for the Gaudin algebra.

6 Quasi-exact solvability of quasi-Gaudin models

Substituting (5.2) into (3.3), we obtain the following explicit form of quasi-Gaudin hamiltonians:

$$\begin{aligned}
H_n(\lambda) &= S^0(\lambda)S^0(\lambda) - \frac{1}{2}S^-(\lambda)S^+(\lambda) - \frac{1}{2}S^+(\lambda)S^-(\lambda) + \\
&+ \frac{2S^0(\lambda)(n+b+F^0-S^0) - S^-(\lambda)(n+2b+F^0-S^0) - S^+(\lambda)(n+F^0-S^0)}{\lambda-a} + \\
&+ \frac{b(b-1)}{(\lambda-a)^2}
\end{aligned} \tag{6.1}$$

Now we can check in an independent way that the operators (6.1) actually describe quasi-exactly solvable models. Denote by Φ_n a linear hull of all vectors $S^+(\xi_k)\dots S^+(\xi_1)|0\rangle$ with arbitrary ξ_1, \dots, ξ_k and $k \leq n$. It is known that $\dim \Phi_n < \infty$ for any n if $F^0(\lambda)$ is a rational function. From the obvious relations $S^\pm(\lambda)\Phi_n \subset \Phi_{n\pm 1}$, $S^0(\lambda)\Phi_n \subset \Phi_n$, $(n+F^0-S^0)\Phi_n \subset \Phi_{n-1}$ and $(n+F^0-S^0)\Phi_m \subset \Phi_m$ for $m \neq n$ it immediately follows that each of the operators $H_n(\lambda)$ admits only one invariant subspace, Φ_n . This subspace is finite-dimensional and therefore the models (6.1) are quasi-exactly solvable in the standard sense of this word (for the “standard” quasi-exact solvability see e.g. refs. [25, 23, 26]).

7 Quasi- $sl(2)$ algebra

It is known that the $sl(2)$ algebra can be considered as a limiting case of the Gaudin $\mathcal{G}[sl(2)]$ algebra (2.1). Its generators S^0 , S^- and S^+ are given by

$$S^0 = \lim_{\lambda \rightarrow \infty} \lambda S^0(\lambda), \quad S^- = \lim_{\lambda \rightarrow \infty} \lambda S^-(\lambda), \quad S^+ = \lim_{\lambda \rightarrow \infty} \lambda S^+(\lambda), \tag{7.1}$$

and satisfy the usual commutation relations

$$\begin{aligned}
S^0 S^- - S^- S^0 &= -S^-, \\
S^0 S^+ - S^+ S^0 &= +S^+, \\
S^- S^+ - S^+ S^- &= 2S^0.
\end{aligned} \tag{7.2}$$

It is natural to ask what kind of algebra will appear if we consider an analogous limit of quasi- $\mathcal{G}[sl(2)]$ algebra? By analogy with the Gaudin case, we can define the generators of this algebra by the formulas

$$S_n^0 = \lim_{\lambda \rightarrow \infty} \lambda S_n^0(\lambda), \quad S_n^- = \lim_{\lambda \rightarrow \infty} \lambda S_n^-(\lambda), \quad S_n^+ = \lim_{\lambda \rightarrow \infty} \lambda S_n^+(\lambda). \tag{7.3}$$

Multiplying the quasi-commutation relations (3.1) by $\lambda\mu$ and tending both λ and μ to infinity we easily derive the relations between these generators, which read

$$\begin{aligned}
S_{n-1}^0 S_n^- - S_n^- S_{n-1}^0 &= -S_n^-, \\
S_{n+1}^0 S_n^+ - S_n^+ S_{n+1}^0 &= +S_n^+, \\
S_{n+1}^- S_n^+ - S_n^+ S_{n+1}^- &= 2S_n^0.
\end{aligned} \tag{7.4}$$

We can consider (7.4) as the defining relations of a certain modification of the $sl(2)$ algebra which can be called “quasi- $sl(2)$ algebra”. Despite the fact that it *is not* a Lie algebra, it has many properties similar to those of the ordinary $sl(2)$. For example, it has a quasi-analogue of the Casimir operator,

$$C_n = S_n^0 S_n^0 - \frac{1}{2} S_{n+1}^- S_n^+ - \frac{1}{2} S_{n-1}^+ S_n^- \quad (7.5)$$

which quasi-commutes with all generators:

$$C_n S_{n-1}^+ = S_{n-1}^+ C_{n-1}, \quad C_n S_{n+1}^- = S_{n+1}^- C_{n+1}, \quad C_n S_n^0 = S_n^0 C_n. \quad (7.6)$$

The “representations” of quasi- $sl(2)$ algebra can be constructed in the same way as in the quasi-Gaudin case. Defining the “lowest weight vector” $|0\rangle$ by

$$S_0^0 |0\rangle = F |0\rangle, \quad S_0^- |0\rangle = 0, \quad (7.7)$$

where F is a certain arbitrarily fixed number, we can define the “representation space” W_F as a linear hull of vectors

$$|m\rangle_k = S_{m+k}^+ S_{m-1+k}^+ \cdots S_{2+k}^+ S_{1+k}^+ |0\rangle, \quad (7.8)$$

with arbitrary $k \in \mathbb{Z}$ and $m \in \mathbb{N}$. If we are interested in the spectrum of quasi-Casimir operator, then it is easily seen that, in contrast with the standard $sl(2)$ case, it is not infinitely-degenerate and contains only one exactly constructable eigenvector. For example, the only possibility for vector (7.8) to be an eigenvector of (7.5) is realized when $k = -1$ and $m = n$. In this case, the corresponding eigenvalue is equal to $F(F - 1)$. So, one can say that the quasi-Casimir operators represent the simplest (and, in some sense, trivial) quasi-exactly solvable models.

It is remarkable that the generators of quasi- $sl(2)$ algebra can be realized in terms of the generators of ordinary $sl(2)$ algebra. The corresponding formulas can be obtained after substituting formulas (5.2) into (7.3) and read

$$S_n^- = S^- + n + F^0 - S^0, \quad S_n^0 = n + b + F^0, \quad S_n^+ = S^+ + n + 2b + F^0 - S^0. \quad (7.9)$$

This realization corresponds to the choice $F = F^0 + b$ in (7.7).

8 The Yangian and the XXX model

Let us give some brief exposition of facts concerning the Yangian $\mathcal{Y}[sl(2)]$, its representations and properties of the so-called XXX models associated with it. A more detailed exposition of this subject can be found in refs. [6, 20]. The commutation relations for generators $T_{\alpha\beta}(\lambda)$, $\alpha, \beta = 1, 2$ can be extracted from formulas (1.1) and (1.6). Introducing the commonly used notations

$$T_{11}(\lambda) = A(\lambda), \quad T_{12}(\lambda) = B(\lambda), \quad T_{21}(\lambda) = C(\lambda), \quad T_{22}(\lambda) = D(\lambda), \quad (8.1)$$

we can rewrite these relations in more explicit form

$$\begin{aligned}
(\lambda - \mu)\{A(\lambda)B(\mu) - B(\mu)A(\lambda)\} &= \eta\{A(\mu)B(\lambda) - A(\lambda)B(\mu)\} \\
(\lambda - \mu)\{A(\lambda)C(\mu) - C(\mu)A(\lambda)\} &= \eta\{A(\lambda)C(\mu) - A(\mu)C(\lambda)\} \\
(\lambda - \mu)\{D(\lambda)B(\mu) - B(\mu)D(\lambda)\} &= \eta\{D(\lambda)B(\mu) - D(\mu)B(\lambda)\} \\
(\lambda - \mu)\{D(\lambda)C(\mu) - C(\mu)D(\lambda)\} &= \eta\{D(\mu)C(\lambda) - D(\lambda)C(\mu)\} \\
(\lambda - \mu)\{A(\lambda)D(\mu) - D(\mu)A(\lambda)\} &= \eta\{C(\mu)B(\lambda) - C(\lambda)B(\mu)\} \\
(\lambda - \mu)\{B(\lambda)C(\mu) - C(\mu)D(\lambda)\} &= \eta\{D(\mu)A(\lambda) - D(\lambda)A(\mu)\}.
\end{aligned} \tag{8.2}$$

Using these relations one can easily prove that the operators

$$H(\lambda) = A(\lambda) + D(\lambda) \tag{8.3}$$

commute with each other,

$$[H(\lambda), H(\mu)] = 0, \tag{8.4}$$

for any λ and μ . This enables one to consider them as integrals of motion of a completely integrable quantum system which is known under name of XXX model¹⁵. The hamiltonians $H(\lambda)$ of these models act in the representation space of algebra $\mathcal{Y}[sl(2)]$ which is defined as a linear hull of vectors

$$|\xi_1, \dots, \xi_n\rangle = C(\xi_n) \dots C(\xi_1)|0\rangle \tag{8.5}$$

with arbitrary n and ξ_1, \dots, ξ_n . Here $|0\rangle$ is the lowest weight vector, satisfying the conditions

$$B(\lambda)|0\rangle = 0, \quad A(\lambda)|0\rangle = a(\lambda)|0\rangle, \quad D(\lambda)|0\rangle = d(\lambda)|0\rangle, \tag{8.6}$$

and functions $a(\lambda)$ and $b(\lambda)$ play the role of lowest weight functions determining the form of the representation¹⁶. Denoting the representation space defined by formulas (8.5) and (8.6) by $W_{a(\lambda), d(\lambda)}$, we can write down the spectral problem for $H(\lambda)$

$$H(\lambda)\phi = E(\lambda)\phi, \quad \phi \in W_{a(\lambda), d(\lambda)}. \tag{8.7}$$

It turns out that this problem is exactly solvable within the Bethe ansatz

$$\phi = C(\xi_n) \dots C(\xi_1)|0\rangle. \tag{8.8}$$

This can be demonstrated exactly as in the Gaudin case. We simply should act on (8.8) by operator $H(\lambda)$ and transfer the operators $A(\lambda)$ and $D(\lambda)$ (from which it consists) to the right by using bilinear relations

$$\begin{aligned}
A(\lambda)C(\xi_i) &= \frac{-\eta}{\lambda - \xi_i}C(\lambda)A(\xi_i) + \frac{\lambda - \xi_i + \eta}{\lambda - \xi_i}C(\xi_i)A(\lambda), \\
D(\lambda)C(\xi_i) &= \frac{+\eta}{\lambda - \xi_i}C(\lambda)D(\xi_i) + \frac{\lambda - \xi_i - \eta}{\lambda - \xi_i}C(\xi_i)D(\lambda),
\end{aligned} \tag{8.9}$$

¹⁵More precisely — the XXX inhomogeneous magnetic chain.

¹⁶Some formulas of this section differ a little bit from those used in the standard literature since, instead of the standardly used highest weight representation we are considering here the lowest weight one. But the results are, of course, the same.

which trivially follow from (8.2). As soon as operators $A(\lambda), D(\lambda)$ (as well as operators $A(\xi_i), D(\xi_i)$, $i = 1, \dots, n$ arising after these transferences) appear in front of the lowest weight vector $|0\rangle$, they should be absorbed by it. The Bethe ansatz equations guaranteeing the absence of the unwanted terms have the form

$$\prod_{k=1, k \neq i}^n \frac{\xi_i - \xi_k - \eta}{\xi_i - \xi_k + \eta} = \frac{a(\xi_i)}{d(\xi_i)}, \quad i = 1, \dots, n. \quad (8.10)$$

and the final expression for the spectrum of the XXX model reads

$$E(\lambda) = a(\lambda) \prod_{i=1}^n \frac{\lambda - \xi_i + \eta}{\lambda - \xi_i} + d(\lambda) \prod_{i=1}^n \frac{\lambda - \xi_i - \eta}{\lambda - \xi_i}. \quad (8.11)$$

If, for example, $a(\lambda)$ and $d(\lambda)$ are rational functions, then the number of solutions of Bethe ansatz equations (8.10) is finite for any n . But since n is an *arbitrary* non-negative integer, the complete number of solutions of problem (8.7) is infinite. It can be shown that these solutions expire all possible solutions of problem (8.7) and therefore it is exactly solvable.

9 Quasi-Yangian and quasi- XXX models

Let us now construct the quasi-Yangian. Its generators depend additionally on an integer n and can be denoted by $A_n(\lambda)$, $B_n(\lambda)$, $C_n(\lambda)$ and $D_n(\lambda)$. The quasi-commutation relations for these generators read

$$\begin{aligned} (\lambda - \mu)\{A_{n-1}(\lambda)B_n(\mu) - B_n(\mu)A_n(\lambda)\} &= \eta\{A_{n-1}(\mu)B_n(\lambda) - A_{n-1}(\lambda)B_n(\mu)\} \\ (\lambda - \mu)\{A_{n+1}(\lambda)C_n(\mu) - C_n(\mu)A_n(\lambda)\} &= \eta\{A_{n+1}(\lambda)C_n(\mu) - A_{n+1}(\mu)C_n(\lambda)\} \\ (\lambda - \mu)\{D_{n-1}(\lambda)B_n(\mu) - B_n(\mu)D_n(\lambda)\} &= \eta\{D_{n-1}(\lambda)B_n(\mu) - D_{n-1}(\mu)B_n(\lambda)\} \\ (\lambda - \mu)\{D_{n+1}(\lambda)C_n(\mu) - C_n(\mu)D_n(\lambda)\} &= \eta\{D_{n+1}(\mu)C_n(\lambda) - D_{n+1}(\lambda)C_n(\mu)\} \\ (\lambda - \mu)\{A_n(\lambda)D_n(\mu) - D_n(\mu)A_n(\lambda)\} &= \eta\{C_{n-1}(\mu)B_n(\lambda) - C_{n-1}(\lambda)B_n(\mu)\} \\ (\lambda - \mu)\{B_{n+1}(\lambda)C_n(\mu) - C_{n-1}(\mu)B_n(\lambda)\} &= \eta\{D_n(\mu)A_n(\lambda) - D_n(\lambda)A_n(\mu)\}. \end{aligned} \quad (9.1)$$

It is not difficult to check that these relations are invariant under “quasi-similarity” transformations

$$\begin{aligned} A_n(\lambda) &\rightarrow U_n A_n(\lambda) U_n^{-1}, \\ B_n(\lambda) &\rightarrow U_{n-1} B_n(\lambda) U_n^{-1}, \\ C_n(\lambda) &\rightarrow U_{n+1} C_n(\lambda) U_n^{-1}, \\ D_n(\lambda) &\rightarrow U_n D_n(\lambda) U_n^{-1}, \end{aligned} \quad (9.2)$$

in which U_n are arbitrary invertible operators.

Using (9.1) one can easily prove that the operators

$$H_n(\lambda) = A_n(\lambda) + D_n(\lambda) \quad (9.3)$$

commute with each other,

$$[H_n(\lambda), H_n(\mu)] = 0, \quad (9.4)$$

for any λ and μ and any given n . However, the H -operators with different indices n do not generally commute with each other. This leads us to a conclusion that we deal with an infinite series of *different* completely integrable quantum systems which we shall call the quasi- XXX -models.

In order to formulate a spectral problem for hamiltonians $H_n(\lambda)$ one should construct a carrier space in which they act. This space, as usually, can be identified with a “representation space” of quasi- $\mathcal{V}[sl(2)]$ algebra. For constructing the latter we can use the same reasonings as in section 3 and define it as a linear hull of vectors

$$|\xi_1, \dots, \xi_m\rangle_k = C_{m+k}(\xi_m)C_{m+k-1}(\xi_{m-1}) \dots C_{k+2}(\xi_2)C_{k+1}(\xi_1)|0\rangle \quad (9.5)$$

with arbitrary $m \in N$, $k \in Z$ and ξ_1, \dots, ξ_m . It is not difficult to see that, due to commutation relations (9.1), the vectors (9.5) are symmetric with respect to all permutations of numbers ξ_1, \dots, ξ_m . Here, as before, $|0\rangle$ is the lowest weight vector, satisfying the restricted conditions

$$B_0(\lambda)|0\rangle = 0, \quad A_0(\lambda)|0\rangle = a(\lambda)|0\rangle, \quad D_0(\lambda)|0\rangle = d(\lambda)|0\rangle, \quad (9.6)$$

and functions $a(\lambda)$ and $d(\lambda)$ play the role of the lowest weight functions determining the form of the “representation”. We denote such a “representation space” by $W_{a(\lambda), d(\lambda)}$. In the next section we demonstrate that the lowest weight vector is actually not annihilated by other lowering operators $B_n(\lambda)$ with $n \neq 0$. Moreover, we show that it is even not an eigenvector of operators $D_n(\lambda)$ if $n \neq 0$. In conclusion we present the final form of the spectral problem for operators $H_n(\lambda)$, which reads

$$H_n(\lambda)\phi = E(\lambda)\phi, \quad \phi \in W_{a(\lambda), d(\lambda)} \quad (9.7)$$

and whose solutions will be discussed in the next section.

10 Bethe ansatz for quasi- XXX model

The results of section 9 enable one to show that spectral problems (9.7) can be algebraically solved within the Bethe ansatz

$$\phi = C_{m+k}(\xi_m)C_{m+k-1}(\xi_{m-1}) \dots C_{k+2}(\xi_2)C_{k+1}(\xi_1)|0\rangle \quad (10.1)$$

only if the numbers $m \in N$ and $k \in Z$ are chosen as $m = n$ and $k = -1$, where n is the index of the hamiltonian. In this case (10.1) takes the form

$$\phi = C_{n-1}(\xi_n)C_{n-2}(\xi_{n-1}) \dots C_1(\xi_2)C_0(\xi_1)|0\rangle, \quad (10.2)$$

and we can use the “quasi” analogues of relations (8.9)

$$\begin{aligned} A_n(\lambda)C_{n-1}(\xi_n) &= \frac{-\eta}{\lambda - \xi_n}C_{n-1}(\lambda)A_{n-1}(\xi_n) + \frac{\lambda - \xi_n + \eta}{\lambda - \xi_n}C_{n-1}(\xi_n)A_{n-1}(\lambda), \\ D_n(\lambda)C_{n-1}(\xi_n) &= \frac{+\eta}{\lambda - \xi_n}C_{n-1}(\lambda)D_{n-1}(\xi_n) + \frac{\lambda - \xi_n - \eta}{\lambda - \xi_n}C_{n-1}(\xi_n)D_{n-1}(\lambda) \end{aligned} \quad (10.3)$$

(which can be easily obtained from basic relations (9.1)) to transfer the operators $A_n(\lambda)$ and $D_n(\lambda)$ to the right. Exactly as in the case of quasi-Gaudin algebra, each permutation of A -

or D -operators with C -operators forming Bethe vector decreases the indices of the former by one. Finally, when all permutations will be completed and the operators $A_n(\lambda), D_n(\lambda)$ together with the daughter operators $A_n(\xi_i), D_n(\xi_i)$, $\xi = 1, \dots, n$ appear in front of the lowest weight vector $|0\rangle$, they will have index 0 and therefore can be absorbed by this vector by means of formulas (9.6). The condition of the cancellation of unwanted terms leads to the system of Bethe ansatz equations

$$\prod_{k=1, k \neq i}^n \frac{\xi_i - \xi_k - \eta}{\xi_i - \xi_k + \eta} = \frac{a(\xi_i)}{d(\xi_i)}, \quad i = 1, \dots, n, \quad (10.4)$$

whose form exactly coincides with (8.10), and the corresponding eigenvalues read

$$E(\lambda) = a(\lambda) \prod_{i=1}^n \frac{\lambda - \xi_i + \eta}{\lambda - \xi_i} + d(\lambda) \prod_{i=1}^n \frac{\lambda - \xi_i - \eta}{\lambda - \xi_i}, \quad (10.5)$$

which is again exactly the same expression as in (8.11). Summarising, we can say, that the use of quasi- $\mathcal{Y}[sl(2)]$ algebra enables one to construct an infinite set of completely integrable systems, each of which is only partially solvable and has only a certain finite number of solutions (of course, provided that both $a(\lambda)$ and $b(\lambda)$ are rational functions). The second interesting fact is that the we obtained exactly the same set of solutions as in the case of ordinary Yangian. But now these solutions are distributed between different models.

11 The existence of quasi-Yangian

In this section we present one of possible realizations of quasi-Yangian. In full accordance with the Gaudin case, we construct it from generators of the ordinary Yangian. Assume that the latter is characterized by four generators $A(\lambda), B(\lambda), C(\lambda), D(\lambda)$ acting in a certain representation space with lowest weights $a^0(\lambda)$ and $d^0(\lambda)$. Introduce the limiting operator

$$S^0 = \frac{1}{2\eta} \lim_{\lambda \rightarrow \infty} \lambda(A(\lambda) - D(\lambda)) \quad (11.1)$$

and its lowest weight

$$F^0 = \frac{1}{2\eta} \lim_{\lambda \rightarrow \infty} \lambda(a^0(\lambda) - d^0(\lambda)). \quad (11.2)$$

Then the needed realization reads

$$\begin{aligned} A_n(\lambda) &= A(\lambda) \left(1 + \eta \frac{F^0 - S^0 + n - 1/2 + b}{\lambda - a} \right) + B(\lambda) \left(\eta \frac{F^0 - S^0 + n + 2b}{\lambda - a} \right), \\ B_n(\lambda) &= B(\lambda) \left(1 - \eta \frac{F^0 - S^0 + n + 1/2 + b}{\lambda - a} \right) - A(\lambda) \left(\eta \frac{F^0 - S^0 + n}{\lambda - a} \right), \\ C_n(\lambda) &= C(\lambda) \left(1 + \eta \frac{F^0 - S^0 + n - 1/2 + b}{\lambda - a} \right) + D(\lambda) \left(\eta \frac{F^0 - S^0 + n + 2b}{\lambda - a} \right), \\ D_n(\lambda) &= D(\lambda) \left(1 - \eta \frac{F^0 - S^0 + n + 1/2 + b}{\lambda - a} \right) - C(\lambda) \left(\eta \frac{F^0 - S^0 + n}{\lambda - a} \right). \end{aligned} \quad (11.3)$$

One can easily check by direct calculations that these operators actually satisfy the quasi-commutation relations (9.1) and thus form the quasi- $\mathcal{Y}[sl(2)]$. Since the representation space of this quasi-Yangian coincides by construction with the representation space of the ordinary Yangian, we can check formulas (9.6) by using their Yangian analogues (8.6). Taking into account that

$$S^0|0\rangle = F^0|0\rangle, \quad (11.4)$$

and using formulas (8.6), we obtain:

$$\begin{aligned} A_n(\lambda)|0\rangle &= a^0(\lambda) \left(1 + \eta \frac{n - 1/2 + b}{\lambda - a}\right) |0\rangle, \\ D_n(\lambda)|0\rangle &= d^0(\lambda) \left(1 - \eta \frac{n + 1/2 + b}{\lambda - a}\right) |0\rangle - \left(\eta \frac{n}{\lambda - a}\right) C(\lambda)|0\rangle, \\ B_n(\lambda)|0\rangle &= -a^0(\lambda) \left(\eta \frac{n}{\lambda - a}\right) |0\rangle. \end{aligned} \quad (11.5)$$

We see that the lowest weight vector is actually annihilated by “lowering” operators $B_n(\lambda)$ and is an eigenvalue of operators $D_n(\lambda)$ only for $n = 0$. Taking $n = 0$ in (11.5) we obtain the expressions for the lowest weights of quasi-Yangian

$$a(\lambda) = a^0(\lambda) \left(1 + \eta \frac{b - 1/2}{\lambda - a}\right), \quad d(\lambda) = d^0(\lambda) \left(1 - \eta \frac{b + 1/2}{\lambda - a}\right). \quad (11.6)$$

Now it is absolutely clear that algebra (9.1) is a continuous deformation of the Yangian (8.2). The role of the deformation parameter is played by a . If $a \rightarrow \infty$, then the n -dependence of the operators (11.3) disappears and they transform into the ordinary generators of Yangian. Respectively, the quasi-relations in (9.1) become the ordinary ones. As to the formulas (9.6) defining the representations of algebra (9.1), they, in the limit $a \rightarrow \infty$ also transform into the relations (8.6) for the Yangian.

12 Quasi-exact solvability of quasi-XXX models

Substituting (11.3) into (9.3), we obtain an explicit form of hamiltonians of quasi-XXX models:

$$\begin{aligned} H_n(\lambda) &= \frac{\lambda - a - \eta}{\lambda - a} (A(\lambda) + D(\lambda)) \\ &+ \frac{\eta}{\lambda - a} (A(\lambda) - D(\lambda)) (F^0 - S^0 + n + b) + \\ &+ \frac{\eta}{\lambda - a} B(\lambda) (F^0 - S^0 + n + 2b) - \\ &- \frac{\eta}{\lambda - a} C(\lambda) (F^0 - S^0 + n). \end{aligned} \quad (12.1)$$

Let us now check in an independent way that the operators (12.1) actually describe quasi-exactly solvable models. Denote by Φ_n the linear hull of all vectors $C(\xi_k) \dots C(\xi_1)|0\rangle$ with arbitrary ξ_1, \dots, ξ_k and $k \leq n$. It is known that $\dim \Phi_n < \infty$ for any n if $a^0(\lambda)$ and $d^0(\lambda)$ are rational functions. From the obvious relations $B(\lambda)\Phi_n \subset \Phi_{n-1}$, $C(\lambda)\Phi_n \subset \Phi_{n+1}$,

$A(\lambda)\Phi_n \subset \Phi_n$, $D(\lambda)\Phi_n \subset \Phi_n$, $(F^0 - S^0 + n)\Phi_n \subset \Phi_{n-1}$ and $(F^0 - S^0 + n)\Phi_m \subset \Phi_m$ for $m \neq n$ it immediately follows that each of the operators $H_n(\lambda)$ admits only one algebraically constructable invariant subspace, Φ_n . This subspace is finite-dimensional and therefore the models (12.1) are quasi-exactly solvable.

13 Historical roots

Quasi-exactly solvable models came to the light about ten years ago. After first examples which have been found heuristically [54, 1, 36], the three general theories explaining the phenomenon of quasi-exact solvability and presenting constructive ways for building such models have been formulated. These theories are known under names of “partial algebraization method” [37, 25, 23, 14, 15, 38, 26, 16], “the inverse method of separation of variables” [41, 43, 46] and “the projection method” [45, 46]. In sections 14 – 16 we try to interpret the results obtained in previous sections from the point of view of these three methods and show that, in contrast with the first method, which is not applicable to the study of quasi- \mathcal{T}_R algebras at all, the last two ones enable one to reproduce some general features of the proposed formalism. Especially, this relates to the third, projection method, by means of which we essentially obtained all the results exposed in sections 2 – 12.

14 The partial algebraization method

One of the first theories of quasi-exactly solvable systems proposed in 1988 in ref. [37] and called the “partial algebraization method” was based on a purely algebraic construction. Due to simplicity of this theory, its essence can be formulated in several words.

The method

The main idea of the partial algebraization method was based on the observation that any finite-dimensional representation of any Lie or Lie_q algebra can be realized by means of first order differential or pseudo-differential operators acting in the spaces of functions. Let L_i , $i = 1, \dots, d$ be such generators. Taking their combinations of the form

$$H = \sum_{i,k=1}^d C_{ik} L_i L_k + \sum_{i=1}^d C_i L_i \quad (14.1)$$

one obtains a certain second-order differential or pseudo-differential operator having (by construction) a finite-dimensional invariant subspace. This means that for this operator at least a certain finite part of the spectrum can be constructed algebraically (for more detail see refs. [37, 25, 23, 14, 15, 38, 26, 16]). Note also that this approach is easily extendable to the matrix case (see e.g. refs. [25, 4]).

Discussion

The partial algebraization method seems to be rather general. The models obtainable in such a way are not necessarily integrable because only an existence of algebraically constructable finite-dimensional invariant subspaces is required. But just because of the generality of this

method, it contains an essential deficiency not giving a constructive way of solving the resulting algebraic problems. Essentially, the method stops as soon as the representability of a certain interesting hamiltonian in the form (14.1) is proven.

What can we say of quasi-Gaudin and quasi-XXX spin models from the point of view of this theory? Unfortunately, nothing, because neither the generators of the initial \mathcal{T}_R algebra used in formulas (2.3) and (8.3) nor their quasi-counterparts used in (3.3) and (9.3) are assumed to realize any finite-dimensional representation. The problem of finding algebras and their finite-dimensional representations which would allow one to represent our hamiltonians in the form (14.1) is interesting and we hope that it will find its solution in the nearest future.

15 The inverse method of separation of variables

Another oldest theory of quasi-exactly solvable systems proposed in 1988 in ref. [41] and developed in [43, 46] was purely analytic. It can be divided into two parts: 1) construction of one-dimensional exactly- and quasi-exactly solvable multi-parameter spectral equations and 2) reduction of these equations to ordinary one-parameter exactly- or quasi-exactly solvable spectral equations in a multi-dimensional space. Consider this method for the simplest case of second-order equations.

The first part of the method

Let ∂ denote the differential operator, $\partial\phi(\lambda) = \phi'(\lambda)$, and let $F(\lambda)$ be a given rational function. Consider the equation

$$(\partial - F(\lambda))^2 \Psi(\lambda) = E(\lambda) \Psi(\lambda) \quad (15.1)$$

for two functions $E(\lambda)$ and $\Psi(\lambda)$. It can be shown that it admits a very simple class of solutions

$$\Psi(\lambda) = \prod_{i=1}^n (\lambda - \xi_i), \quad n = 0, 1, 2, \dots, \quad (15.2)$$

$$E(\lambda) = F'(\lambda) + F^2(\lambda) + 2 \sum_{i=1}^n \frac{F(\lambda) - F(\xi_i)}{\lambda - \xi_i}, \quad n = 0, 1, 2, \dots, \quad (15.3)$$

where, for any given n , the numbers ξ_1, \dots, ξ_n satisfy the system of numerical equations

$$\sum_{k=1, k \neq i}^n \frac{1}{\xi_i - \xi_k} + F(\xi_i) = 0, \quad i = 1, \dots, n. \quad (15.4)$$

Before discussing the mathematical meaning of the obtained solutions (which obviously form only a very small subset in the set of all possible solutions of equation (15.1)), let us consider an analogous construction for pseudo-differential operators. Let T_η denote the shift operator, $T_\eta\phi(\lambda) = \phi(\lambda + \eta)$, and $a(\lambda), d(\lambda)$ be some rational functions on λ . Consider the equation

$$(a(\lambda)T_\eta + d(\lambda)T_\eta^{-1})\Psi(\lambda) = E(\lambda)\Psi(\lambda) \quad (15.5)$$

for two functions $E(\lambda)$ and $\Psi(\lambda)$. It can be shown that it admits a very simple class of solutions

$$\Psi(\lambda) = \prod_{i=1}^n (\lambda - \xi_i), \quad n = 0, 1, 2, \dots, \quad (15.6)$$

$$E(\lambda) = a(\lambda) \prod_{i=1}^n \frac{\lambda - \xi_i + \eta}{\lambda - \xi_i} + d(\lambda) \prod_{i=1}^n \frac{\lambda - \xi_i - \eta}{\lambda - \xi_i}, \quad n = 0, 1, 2, \dots, \quad (15.7)$$

where, for any given n , the numbers ξ_1, \dots, ξ_n satisfy the system of numerical equations

$$\prod_{k=1, k \neq i}^n \frac{\xi_i - \xi_k - \eta}{\xi_i - \xi_k + \eta} = \frac{a(\xi_i)}{d(\xi_i)}, \quad i = 1, \dots, n, \quad (15.8)$$

An analogous construction for pseudo-differential equations of a little bit different type was considered recently in ref. [49].

Now let us explain the reason for which these solutions are interesting to us. A remarkable fact (distinguishing these solutions from an infinite number of other solutions) is that for any rational function $F(\lambda)$ (in differential case) and for any pair of rational functions $a(\lambda)$ and $d(\lambda)$ (in pseudo-differential case) the function $E(\lambda)$ is representable in the form

$$E(\lambda) = W_0(\lambda) + \sum_{\alpha=1}^N E_\alpha W_\alpha(\lambda), \quad (15.9)$$

where N is some number depending only on the form of function $F(\lambda)$ or, respectively, of $a(\lambda)$ and $d(\lambda)$, and $W_n(\lambda)$ are fixed functions not depending on ξ_i . All the dependence of $E(\lambda)$ on numbers ξ_i satisfying the equations (15.4) or (15.8) is concentrated in the numerical coefficients E_α . It can be shown that for any given n both equation (15.4) and (15.8) have exactly $(N - 1 + n)! / ((N - 1)! n!)$ solutions. Since n in formulas (15.2), (15.3) and (15.6), (15.7) can be made arbitrarily large, a total number of algebraic solutions of equations (15.1) and (15.5) is infinite. These reasonings enable one to interpret (15.1) and (15.5) as a certain exactly (algebraically) solvable N -parameter spectral equation. Indeed, introducing the notation

$$D(\lambda) = \begin{cases} (\partial - F(\lambda))^2 - W_0(\lambda), & \text{in differential case,} \\ a(\lambda)T_\eta + d(\lambda)T_\eta^{-1} - W_0(\lambda), & \text{in pseudo-differential case,} \end{cases} \quad (15.10)$$

we can rewrite the equations (15.1) and (15.5) in the following common form

$$D(\lambda)\Psi(\lambda) = \left(\sum_{\alpha=1}^N E_\alpha W_\alpha(\lambda) \right) \Psi(\lambda). \quad (15.11)$$

In particular case, when $N = 1$, (15.11) reduces to an ordinary one-parameter spectral equation which, in turn, can easily be reduced to the Schrödinger form. This gives us the list of known one-dimensional exactly solvable models.

A remarkable feature of equation (15.11) is that the spectrum of one of its spectral parameters is always degenerate, i.e. depends only on n but not on the numbers ξ_i . Assume

for definiteness that the parameter E_N has such a degenerate spectrum. Then, introducing the notation $E_N = E_{n,N}$ and taking

$$D_n(\lambda) = \begin{cases} (\partial - F(\lambda))^2 - W_0(\lambda) - E_{n,N}W_N(\lambda), & \text{in differential case,} \\ a(\lambda)T_\eta + d(\lambda)T_\eta^{-1} - W_0(\lambda) - E_{n,N}W_N(\lambda), & \text{in pseudo-differential case,} \end{cases} \quad (15.12)$$

we can represent (15.11) as an infinite sequence of $(N-1)$ -parameter spectral equations

$$D_n(\lambda)\Psi(\lambda) = \left(\sum_{\alpha=1}^{N-1} E_\alpha W_\alpha(\lambda) \right) \Psi(\lambda), \quad n = 0, 1, 2, \dots \quad (15.13)$$

each of which will obviously be only quasi-exactly solvable. For $N = 2$ these equations can again be considered as one-parameter spectral equations easily reducible to the Schrödinger form. This gives us the list of all known series of one-dimensional quasi-exactly solvable models. This completes the first part of the method.

The second part of the method

Now, what to do if $N > 1$ and $N > 2$ in the first and second cases, respectively? Remember that any one-dimensional d -parameter spectral equation with spectral parameters E_1, \dots, E_d can be interpreted as a result of separation of variables in a certain d -dimensional separable (and hence integrable) system. The parameters E_1, \dots, E_d play in this case the role of separation constants and their admissible values have the meaning of the eigenvalues of some commuting d -dimensional operators — the integrals of motion of the abovementioned system. Note that the form of these integrals can easily be reconstructed by means of the so-called inverse procedure of separation of variables.

Let us first find the N integrals of motion, H_1, \dots, H_N for the first equation (15.11) and, after this, construct their combination

$$H(\lambda) = W_0(\lambda) + \sum_{\alpha=1}^N H_\alpha W_\alpha(\lambda). \quad (15.14)$$

The operators $H(\lambda)$ will obviously commute with each other,

$$[H(\lambda), H(\mu)] = 0, \quad (15.15)$$

and will have the eigenvalues given by formulas (15.3) or (15.7). This means that the system described by these operators will not be only integrable but also exactly solvable.

Let us now find the integrals of motion associated with the second system of multi-parameter spectral equations (15.13). Since the form of these equations depends explicitly on n , the obtained integrals also should depend on n . Denoting them by $H_{n,1}, \dots, H_{n,N-1}$, let us construct their analogous combinations

$$H_n(\lambda) = W_0(\lambda) + E_{n,N}W_N(\lambda) + \sum_{\alpha=1}^{N-1} H_{n,\alpha}W_\alpha(\lambda), \quad n = 0, 1, 2, \dots \quad (15.16)$$

From this formula it immediately follows that operators $H_n(\lambda)$ form again commutative families

$$[H_n(\lambda), H_n(\mu)] = 0, \quad (15.17)$$

for any n , but they are not obliged to commute with each other if their indices do not coincide. The eigenvalues of operators $H_n(\lambda)$ will be described by the same formulas (15.3) or (15.7) but with fixed n . Therefore all the integrable models represented by these operators will have only a finite number of explicitly constructible solutions and thus will be quasi-exactly solvable. This completes the second part of the method.

Discussion

Now what can we say of this method from the point of view of results obtained in sections 2 – 12? A little bit more than in the case of partial algebraization method. First of all, we arrived to the same situation as in these sections. The quasi-exactly solvable models have one and the same set of solutions as the exactly solvable ones but these solutions are distributed between different models. Second, the spectra (15.3) (resp. (15.7)) of our models and the form of auxiliary conditions (15.4) (resp. (15.8)) for numbers ξ_i exactly coincide with the spectra and Bethe ansatz equations for the Gaudin (resp. XXX) and quasi-Gaudin (resp. quasi- XXX) models. This suggests that we obtained the same models but in a very special coordinate form.

Summarizing, one can say that the inverse method of separation of variables correctly reproduces some external features of models discussed in sections 2 – 12. However, and this is quite obvious, the analytic formalism exposed in this section is not convenient for investigating the properties of the underlying $\mathcal{G}[sl(2)]$ (resp. $\mathcal{Y}[sl(2)]$) and quasi- $\mathcal{G}[sl(2)]$ (resp. quasi- $\mathcal{Y}[sl(2)]$) algebras. Moreover, it is applicable only to the study of separable models and thus cannot pretend on generality.

16 The projection method

The third method of building quasi-exactly solvable models proposed in 1992 in ref. [45] and developed in [46] is the so-called projection method. This method is applicable to all integrable and exactly solvable systems with global symmetry. The idea of this method is very simple and, in fact, it is implicitly encoded in the inverse method of separation of variables which we discussed in the previous section. Indeed, as it follows from the above discussion, the complete separation of variables in both N -dimensional exactly solvable and $(N - 1)$ -dimensional quasi-exactly solvable models (constructable in the framework of the inverse method) leads (by construction) to one and the same multi-parameter spectral equation. But this suggests that the $(N - 1)$ -dimensional quasi-exactly solvable models can be obtained from their N -dimensional exactly solvable counterparts by means of the partial separation of variables. Why this observation is so important for us? Because it gives us a possibility of formulating a new more general method for constructing quasi-exactly solvable models. The generality of this method follows from the fact that it does not require anymore a complete separability of variables in the initial exactly solvable model. The only thing which one needs indeed is the requirement of a partial separability which, roughly speaking, can always be satisfied if the hamiltonian of an initial exactly solvable model admits some global Lie symmetry. Let us now explain in more detail what we mean.

The method

Let $H(\lambda)$ be some quantum hamiltonian acting in a certain linear vector space Φ , and let the spectral problem

$$H(\lambda)\phi = E(\lambda)\phi, \quad \phi \in \Phi \quad (16.1)$$

have an infinite and discrete set of exactly constructable solutions. Assume that it is possible to represent $H(\lambda)$ in the form of a differential or pseudo-differential operator in certain variables x_1, \dots, x_N and consider Φ as a space of polynomials in x_1, \dots, x_N . Obviously, for most of interesting systems this is possible.

Assume now that there exists some operator L which: a) commutes with $H(\lambda)$:

$$[H(\lambda), L] = 0, \quad (16.2)$$

b) is representable in the form of a first-order differential operator in the same variables x_1, \dots, x_N :

$$L = \sum_{i=1}^N A_i(x_1, \dots, x_N) \frac{\partial}{\partial x_i} + B(x_1, \dots, x_N) \quad (16.3)$$

and c) has an infinite and discrete spectrum in Φ . Again, for most of interesting systems such operator does actually exist.

Let l_n , $n = 0, 1, 2, \dots$ and Φ_n , $n = 0, 1, 2, \dots$ denote the eigenvalues and corresponding eigensubspaces of the operator L in Φ . It follows from (16.2) that all these eigensubspaces, which, by definition consist of polynomials in x_1, \dots, x_N , are simultaneously the invariant subspaces for $H(\lambda)$. This means that all spectral problems

$$H(\lambda)\phi = E(\lambda)\phi, \quad \phi \in \Phi_n, \quad n = 0, 1, 2, \dots \quad (16.4)$$

are well defined and are exactly solvable.

Let us now denote by Ψ_n the sets of all (not necessarily normalizable) solutions of the equation

$$L\psi = l_n\psi, \quad n = 0, 1, 2, \dots \quad (16.5)$$

Due to the absence of the condition of polynomiality of these solutions, these sets should be wider than Φ_n :

$$\Phi_n \subset \Psi_n, \quad n = 0, 1, 2, \dots, \quad (16.6)$$

but, as before, they will be the invariant subspaces for $H(\lambda)$, and therefore the extended versions of equations (16.4)

$$H(\lambda)\psi = E(\lambda)\psi, \quad \psi \in \Psi_n, \quad n = 0, 1, 2, \dots \quad (16.7)$$

will also be well defined and, formally, will be quasi-exactly solvable. This, however, will not be that kind of quasi-exact solvability which we discussed before and which we want to have.

This will be rather some pathologically trivial case of it, because the hamiltonian $H(\lambda)$ still does know that it should act only in one of the sets Ψ_n and acts equally well in all of them. Practically this means that we cannot interpret the equations (16.7) as spectral equations for *different* models.

In order to improve the situation, one should restrict somehow the action of the hamiltonian $H(\lambda)$ to one of the sets Ψ_n . But for this it is necessary to have a little bit more information of the structure of these sets. We can extract it from the equation (16.5) which, according to (16.3), can be rewritten as

$$\left(\sum_{i=1}^N A_i(x_1, \dots, x_N) \frac{\partial}{\partial x_i} \right) \psi(x_1, \dots, x_N) = \left(l_n - B(x_1, \dots, x_N) \right) \psi(x_1, \dots, x_N). \quad (16.8)$$

The general solution of this equation has the form

$$\psi(x_1, \dots, x_N) = \rho_n(x_1, \dots, x_n) f[\sigma_1(x_1, \dots, x_n), \dots, \sigma_{N-1}(x_1, \dots, x_n)] \quad (16.9)$$

where ρ_n is a certain arbitrarily fixed particular solution of (16.8) and σ_i , $i = 1, \dots, N-1$ are some functionally independent solutions of the auxiliary equation

$$\left(\sum_{i=1}^N A_i(x_1, \dots, x_N) \frac{\partial}{\partial x_i} \right) \sigma(x_1, \dots, x_N) = 0. \quad (16.10)$$

The function $f(\sigma_1, \dots, \sigma_{N-1})$ in (16.9) is arbitrary and therefore all the information of the specific properties of the spaces Ψ_n is concentrated in ρ_n . The set of all functions $f(\sigma_1, \dots, \sigma_{N-1})$, which we denote by F , contains for any given n a certain subset F_n corresponding to explicit solutions of the initial equations (16.4).

Let us now note that, due to the functional independence of N functions

$$\begin{aligned} \rho &= [\rho_n(x_1, \dots, x_n)]^{\frac{1}{n}}, \\ \sigma_i &= \sigma_1(x_1, \dots, x_n), \quad i = 1, \dots, N-1, \end{aligned} \quad (16.11)$$

they, for any given n , can be considered as the new independent variables (instead of x_1, \dots, x_N). Rewriting the hamiltonian $H(\lambda)$ in terms of these new variables and using the obtained expressions (16.9) for functions ψ with given n , we can rewrite equations (16.7) in the form:

$$\tilde{H}_n(\lambda) f(\sigma_1, \dots, \sigma_{N-1}) = E(\lambda) f(\sigma_1, \dots, \sigma_{N-1}), \quad f \in F, \quad n = 0, 1, 2, \dots \quad (16.12)$$

where

$$\tilde{H}_n(\lambda) = \rho_n^{-1} H(\lambda) \rho_n, \quad n = 0, 1, 2, \dots \quad (16.13)$$

Now the only one last step remains. The step which just suggested us to call this method the projection method. Let us look at formula (16.12). We see that the transformed hamiltonians $\tilde{H}_n(\lambda)$ are, by construction, certain differential or pseudo-differential operator in *all* variables ρ and $\sigma_1, \dots, \sigma_{N-1}$. At the same time, they act only on functions on $N-1$ variables

$\sigma_1, \dots, \sigma_{N-1}$ and give again the functions of the same variables. This means that the most general form of these operators is

$$\tilde{H}_n(\lambda) = H_n(\lambda) + H'_n(\lambda) \frac{\partial}{\partial \rho}, \quad n = 0, 1, 2, \dots, \quad (16.14)$$

where $H'_n(\lambda)$ are certain operators in all variables ρ_n and $\sigma_1, \dots, \sigma_N$, and $H_n(\lambda)$ are operators in variables $\sigma_1, \dots, \sigma_N$ only. This enables one to rewrite (16.12) in the following final form

$$H_n(\lambda) f(\sigma_1, \dots, \sigma_{N-1}) = E(\lambda) f(\sigma_1, \dots, \sigma_{N-1}), \quad f \in F, \quad n = 0, 1, 2, \dots \quad (16.15)$$

We see that these equations have the form of ordinary quasi-exactly solvable ones, because the set F contains by construction the subsets F_n in which these equations are exactly solvable. All the information of a quasi-exactly solvable model is encoded now in its hamiltonian which explicitly depends on n .

Discussion

Let us now try to explain the results of sections 2 — 12 from the point of view of the projection method. For this, let us assume that $H(\lambda)$ is a hamiltonian of Gaudin or XXX model expressed respectively via generators of $\mathcal{G}[sl(2)]$ or $\mathcal{Y}[sl(2)]$ algebras by formulas (2.3) or (8.3). It is known that generators of both $\mathcal{G}[sl(2)]$ and $\mathcal{Y}[sl(2)]$ algebras can be constructed from generators $S_\alpha^-, S_\alpha^0, S_\alpha^+, \alpha = 1, \dots, N$ of algebra $sl(2) \otimes \dots \otimes sl(2)$ (N times). The corresponding formulas have the form

$$\begin{pmatrix} S^0(\lambda) & -S^-(\lambda) \\ +S^+(\lambda) & -S^0(\lambda) \end{pmatrix} = \sum_{\alpha=1}^N \begin{pmatrix} S_\alpha^0 & -S_\alpha^- \\ +S_\alpha^+ & -S_\alpha^0 \end{pmatrix} \frac{1}{\lambda - a_\alpha}, \quad (16.16)$$

for Gaudin algebra $\mathcal{G}[sl(2)]$, and

$$\begin{pmatrix} A(\lambda) & B(\lambda) \\ C(\lambda) & D(\lambda) \end{pmatrix} = \prod_{\alpha=1}^N \left[1 + \frac{\eta}{\lambda - a_\alpha} \begin{pmatrix} S_\alpha^0 & -S_\alpha^- \\ +S_\alpha^+ & -S_\alpha^0 \end{pmatrix} \right], \quad (16.17)$$

for the Yangian $\mathcal{Y}[sl(2)]$. Using formulas (2.3) and (8.3), we can respectively express the hamiltonians $H(\lambda)$ of both Gaudin and XXX models via generators $S_\alpha^-, S_\alpha^0, S_\alpha^+, \alpha = 1, \dots, N$. Remember that these generators can be realized as differential operators

$$S_\alpha^- = \frac{\partial}{\partial x_\alpha}, \quad S_\alpha^0 = x_\alpha \frac{\partial}{\partial x_\alpha} + F_\alpha, \quad S_\alpha^+ = x_\alpha^2 \frac{\partial}{\partial x_\alpha} + 2F_\alpha x_\alpha \quad (16.18)$$

acting on polynomials in variables x_1, \dots, x_N . Formulas (16.16) and (16.17) together with (16.18) enable one to represent the hamiltonian $H(\lambda)$ in the form of some differential operator in N variables x_1, \dots, x_N . The space Φ in which this hamiltonian acts becomes respectively the space of polynomials.

Now note that, the hamiltonian $H(\lambda)$ has a global $sl(2)$ symmetry, which, in particular, implies its commutativity with the operator of the z -projection of total spin

$$L = \sum_{\alpha=1}^N S_\alpha^0. \quad (16.19)$$

According to (16.18), the differential form of this operator is

$$L = \sum_{\alpha=1}^N x_{\alpha} \frac{\partial}{\partial x_{\alpha}} + F, \quad (16.20)$$

where $F = F_1 + \dots + F_N$. The eigenvalues of this operator in Φ are given by $l_n = n + F$ and the corresponding eigensubspaces Φ_n are formed by all homogeneous polynomials of degree N .

To construct the spaces Ψ_n , one should find a general solution of the equation

$$\left(\sum_{\alpha=1}^N x_{\alpha} \frac{\partial}{\partial x_{\alpha}} + F \right) \psi(x_1, \dots, x_N) = (n + F) \psi(x_1, \dots, x_N) \quad (16.21)$$

which, obviously, reads

$$\psi(x_1, \dots, x_N) = x_N^n f\left(\frac{x_1}{x_N}, \dots, \frac{x_{N-1}}{x_N}\right). \quad (16.22)$$

The new variables can hence be chosen in the form:

$$\rho = x_N, \quad \sigma_i = \frac{x_i}{x_N}, \quad i = 1, \dots, N-1. \quad (16.23)$$

According to general prescriptions of the projection method, the first (preliminary) step in the reduction procedure lies in rewriting the hamiltonian $H(\lambda)$ in new variables (16.23) and in constructing the homogeneously transformed hamiltonians $\tilde{H}_n(\lambda)$, according to formula (16.13). In our special case this formula gives:

$$\tilde{H}_n(\lambda) = \rho^{-n} H(\lambda) \rho^n. \quad (16.24)$$

Using formulas (2.3) and (8.3), we obtain for $\tilde{H}_n(\lambda)$:

$$\tilde{H}_n(\lambda) = \tilde{S}_n^0(\lambda) \tilde{S}_n^0(\lambda) - \frac{1}{2} \tilde{S}_n^-(\lambda) \tilde{S}_n^+(\lambda) - \frac{1}{2} \tilde{S}_n^+(\lambda) \tilde{S}_n^-(\lambda), \quad (16.25)$$

$$\tilde{S}_n^0(\lambda) = \rho^{-n} S^0(\lambda) \rho^n, \quad \tilde{S}_n^-(\lambda) = \rho^{-n} S^-(\lambda) \rho^n, \quad \tilde{S}_n^+(\lambda) = \rho^{-n} S^+(\lambda) \rho^n, \quad (16.26)$$

for the Gaudin case, and

$$\tilde{H}_n(\lambda) = \tilde{A}_n(\lambda) + \tilde{D}_n(\lambda), \quad (16.27)$$

$$\tilde{A}_n(\lambda) = \rho^{-n} A(\lambda) \rho^n, \quad \tilde{D}_n(\lambda) = \rho^{-n} D(\lambda) \rho^n, \quad (16.28)$$

for the case of Yangian. An explicit computation of operators staying in the right hand sides of formulas (16.25) and (16.27) gives

$$\begin{aligned} \begin{pmatrix} \tilde{S}_n^0(\lambda) & -\tilde{S}_n^-(\lambda) \\ +\tilde{S}_n^+(\lambda) & -\tilde{S}_n^0(\lambda) \end{pmatrix} &= \begin{pmatrix} \bar{S}^0(\lambda) & -\rho^{-1} \bar{S}^-(\lambda) \\ +\rho \bar{S}^+(\lambda) & -\bar{S}^0(\lambda) \end{pmatrix} + \\ &+ \begin{pmatrix} \rho \frac{\partial}{\partial \rho} + F_N + n - \sigma & -\frac{\partial}{\partial \rho} - \frac{n-\sigma}{\rho} \\ \rho^2 \frac{\partial}{\partial \rho} + 2F_N \rho + (n-\sigma) \rho & -\rho \frac{\partial}{\partial \rho} - F_N - (n-\sigma) \end{pmatrix} \frac{1}{\lambda - a_N}, \end{aligned} \quad (16.29)$$

for Gaudin algebra $\mathcal{G}[sl(2)]$, and

$$\begin{pmatrix} \tilde{A}_n(\lambda) & \tilde{B}_n(\lambda) \\ \tilde{C}_n(\lambda) & \tilde{D}_n(\lambda) \end{pmatrix} = \begin{pmatrix} \bar{A}(\lambda) & \rho^{-1}\bar{B}(\lambda) \\ \rho\bar{C}(\lambda) & \bar{D}(\lambda) \end{pmatrix} \times \\ \times \left[1 + \frac{\eta}{\lambda - a_N} \begin{pmatrix} \rho \frac{\partial}{\partial \rho} + F_N + (n - \sigma) & -\frac{\partial}{\partial \rho} - \frac{n - \sigma}{\rho} \\ \rho^2 \frac{\partial}{\partial \rho} + 2F_N \rho + (n - \sigma)\rho & -\rho \frac{\partial}{\partial \rho} - F_N - (n - \sigma) \end{pmatrix} \right], \quad (16.30)$$

for the Yangian $\mathcal{Y}[sl(2)]$. Here

$$\sigma = \sum_{\alpha=1}^{N-1} \sigma_\alpha \frac{\partial}{\partial \sigma_\alpha} \quad (16.31)$$

and the entries of the matrices staying in the right hand sides of (16.29) and (16.30) are given by

$$\begin{pmatrix} \bar{S}^0(\lambda) & -\bar{S}^-(\lambda) \\ +\bar{S}^+(\lambda) & -\bar{S}^0(\lambda) \end{pmatrix} = \sum_{\alpha=1}^{N-1} \begin{pmatrix} \sigma_\alpha \frac{\partial}{\partial \sigma_\alpha} + F_\alpha & -\frac{\partial}{\partial \sigma_\alpha} \\ \sigma_\alpha^2 \frac{\partial}{\partial \sigma_\alpha} + 2F_\alpha \sigma_\alpha & -\sigma_\alpha \frac{\partial}{\partial \sigma_\alpha} - F_\alpha \end{pmatrix} \frac{1}{\lambda - a_\alpha} \quad (16.32)$$

and

$$\begin{pmatrix} \bar{A}(\lambda) & \bar{B}(\lambda) \\ \bar{C}(\lambda) & \bar{D}(\lambda) \end{pmatrix} = \prod_{\alpha=1}^{N-1} \left[1 + \frac{\eta}{\lambda - a_\alpha} \begin{pmatrix} \sigma_\alpha \frac{\partial}{\partial \sigma_\alpha} + F_\alpha & -\frac{\partial}{\partial \sigma_\alpha} \\ \sigma_\alpha^2 \frac{\partial}{\partial \sigma_\alpha} + 2F_\alpha \sigma_\alpha & -\sigma_\alpha \frac{\partial}{\partial \sigma_\alpha} - F_\alpha \end{pmatrix} \right]. \quad (16.33)$$

In order to complete the reduction of the initial hamiltonian $H(\lambda)$ to a sequence of hamiltonians $H_n(\lambda)$ of quasi-exactly solvable models, one should project the operators (16.25) and (16.27) onto a space of functions depending on variables $\sigma_1, \dots, \sigma_{N-1}$ only. After some simple algebra one obtains

$$H_n(\lambda) = S_n^0(\lambda) S_n^0(\lambda) - \frac{1}{2} S_{n+1}^-(\lambda) S_n^+(\lambda) - \frac{1}{2} S_{n-1}^+(\lambda) S_n^-(\lambda) \quad (16.34)$$

with

$$\begin{aligned} S_n^-(\lambda) &= \bar{S}^-(\lambda) + \frac{n - \sigma}{\lambda - a_N}, \\ S_n^0(\lambda) &= \bar{S}^0(\lambda) + \frac{n - \sigma + F_N}{\lambda - a_N}, \\ S_n^+(\lambda) &= \bar{S}^+(\lambda) + \frac{n - \sigma + 2F_N}{\lambda - a_N}, \end{aligned} \quad (16.35)$$

for the Gaudin case, and

$$H_n(\lambda) = A_n(\lambda) + D_n(\lambda) \quad (16.36)$$

with

$$\begin{aligned} A_n(\lambda) &= \bar{A}(\lambda) \left(1 + \eta \frac{n - \sigma - 1/2 + F_N}{\lambda - a_N} \right) + \bar{B}(\lambda) \left(\eta \frac{n - \sigma + 2F_N}{\lambda - a_N} \right), \\ B_n(\lambda) &= \bar{B}(\lambda) \left(1 - \eta \frac{n - \sigma + 1/2 + F_N}{\lambda - a_N} \right) - \bar{A}(\lambda) \left(\eta \frac{n - \sigma}{\lambda - a_N} \right), \\ C_n(\lambda) &= \bar{C}(\lambda) \left(1 + \eta \frac{n - \sigma - 1/2 + F_N}{\lambda - a_N} \right) + \bar{D}(\lambda) \left(\eta \frac{n - \sigma + 2F_N}{\lambda - a_N} \right), \\ D_n(\lambda) &= \bar{D}(\lambda) \left(1 - \eta \frac{n - \sigma + 1/2 + F_N}{\lambda - a_N} \right) - \bar{C}(\lambda) \left(\eta \frac{n - \sigma}{\lambda - a_N} \right). \end{aligned} \quad (16.37)$$

for the case of Yangian.

It is not difficult to see that formulas (16.34) and (16.36) exactly coincide with expressions (3.3) and (9.3) for hamiltonians of quasi-Gaudin and quasi-XXX models. As to the formulas (16.35) and (16.37), they, up to a trivial change of notations, are identical to formulas (5.2) and (11.3) for generators of $\mathcal{G}[sl(2)]$ and $\mathcal{Y}[sl(2)]$ algebras.

Summarizing, we can say that the projection method enables one to reproduce correctly the structure of quasi-Gaudin and quasi-XXX models together with all properties of the underlying $\mathcal{G}[sl(2)]$ and $\mathcal{Y}[sl(2)]$ algebras. However, this method contains so many technicalities and is so much model dependent that its practical usefulness seems rather doubtful. In next sections I will expose another simple method which practically without any calculations allows one to construct quasi-analogue for any \mathcal{T}_R algebra.

17 Graded associative algebras

Now we start the exposition of a general and model independent method for building quasi- \mathcal{T}_R algebras and their “representations”. The idea of this method is based on the observation that both \mathcal{T}_R and quasi- \mathcal{T}_R algebras can be considered as two *different* realizations of one and the same abstract Z^r -graded¹⁷ unital associative algebra A . The difference between the “ordinary” and “quasi” versions of this algebra lies in a different nature of its elements and also in different definition of the corresponding associative product. In this section we remind the reader some general properties of an abstract algebra A realized in an abstract linear vector space V . The structure of the pair A, V is described by the following standard axioms.

Axiom 1. The set A is a union of non-intersecting layers A^n , $n \in Z^r$ each of which forms a linear space over C . If $a \in A^n$ then we say that a has grading n and write this as $g(a) = n$. The summation of elements belonging to different layers is forbidden.

Axiom 2. For any $n, m \in Z^r$ there exists a multiplication map $(A^n, A^m) \rightarrow A^{n+m}$ which puts into correspondence to any pair of elements $a \in A^n$ and $b \in A^m$ their product $ab \in A^{n+m}$. This product is a) associative: $a(bc) = (ab)c$, for any $a \in A^n$, $b \in A^m$, $c \in A^l$ and $n, m, l \in Z^r$, b) distributive: $(a+b)c = ac+bc$, $c(a+b) = ca+cb$ for any $a, b \in A^n$, $c \in A^m$ and $n, m \in Z^r$, c) has a unit $I \in A^0$ such that $Ia = aI = a$ for any $a \in A^n$ and $n \in Z^r$.

Axiom 3. The set V is a union of non-intersecting layers V^n , $n \in Z^r$ each of which forms a linear space over C . If $v \in V^n$ then we say that v has grading n and write this as $g(v) = n$. The summation of elements belonging to different layers is forbidden.

Axiom 4. For any $n, m \in Z^r$ there exists a multiplication map $(A^n, V^m) \rightarrow V^{n+m}$ which puts into correspondence to any pair of elements $a \in A^n$ and $v \in V^m$ their product $av \in V^{n+m}$. This product is a) associative: $a(bv) = (ab)v$, for any $a \in A^n$, $b \in A^m$, $v \in V^l$ and $n, m, l \in Z^r$, b) distributive: $(a+b)v = av+bv$, $a(v+u) = av+au$ for any $a, b \in A^n$, $v, u \in V^m$ and $n, m \in Z^r$, c) has the same unit $I \in A_0$ such that $Iv = vI = v$ for any $v \in V^n$ and $n \in Z^r$.

Any A -algebra contains two important subalgebras. One, which we denote by A^0 , is formed by all zero-graded elements of A . The second one which we denote by A^+ consists

¹⁷Here $Z^r = Z \otimes \dots \otimes Z$ (r times) denotes the set of all r -dimensional vectors with integer components.

of all elements with non-negative gradings. The non-negativity of a grading means that all components of the grading vector $n = (n_1, \dots, n_r)$ are non-negative integers. The complement of A^+ in A , consisting of elements whose gradings contain at least one negative component (we call such gradings quasi-negative), does not form any subalgebra of A . Nevertheless, this complement plays an important role in applications and we shall denote it by A^- .

The space V also contains an important subspace formed by all vectors with nonnegative gradings. We denote it by V^+ . The importance of the space V^+ lies in the fact that it, under some special conditions, takes the meaning of an algebraically constructable representation space of algebra A . In order to realize these conditions one needs a special so-called lowest weight vector $|0\rangle$ with grading zero which is annihilated by any operator from A^- :

$$a^-|0\rangle = 0, \quad \text{for any } a^- \in A^- \quad (17.1)$$

and is an eigenvector of all operators from A^0 :

$$a^0|0\rangle = \epsilon(a^0)|0\rangle, \quad \text{for any } a^0 \in A^0. \quad (17.2)$$

Here $\epsilon(a^0)$ denotes some linear functional of a^0 which is usually called the lowest weight. Once a vector $|0\rangle$ is constructed, one can construct the space V^+ in a purely algebraic way, simply by acting on it by various elements of A^+ :

$$V^+ = \bigcup_{a^+ \in A^+} a^+|0\rangle. \quad (17.3)$$

Note that the operator elements of A -algebras can sometimes be treated as quantum observables and the vectors of the space V^+ — as physical states. In this case the grading of vectors v from V^+ can be considered as a collection of “occupation numbers” for certain elementary excitations and the action of operators from A^+ and A^- on these vectors — as a process of creation and annihilation of excitations. The lowest weight vector takes respectively the meaning of the vacuum state. If the number of excitations is a conserved quantity, then the hamiltonian of a system should necessarily be a zero-graded element of subalgebra A^0 . Fixing the hamiltonian $h^0 \in A^0$, the vacuum state $|0\rangle \in V^0$ and the space V^+ we essentially fix a physical system.

Once a physical system is fixed, one can state the problem of finding all its physical states. This leads us to an eigenvalue problem for hamiltonian h^0 :

$$h^0\phi = \epsilon\phi, \quad \phi \in V^+. \quad (17.4)$$

Note that the knowledge of only general properties of algebra A listed above is, obviously, not sufficient to present an explicit solution of problem (17.4). In order to solve it, we need an information of more specific properties of this algebra which strongly depend on a concrete form of its elements and therefore are not encoded in axioms 1 – 4. In fact, what we actually need, is the existence of special bilinear relations between the elements of algebra A .

Indeed, assume that for given $h^0 \in A^0$ and each $m \geq 0$ there exists an element $a_m^+ \in V^+$ with grading m for which the relations

$$h^0 \cdot a_m^+ = a_m^+ \cdot b_m^0 + \sum_i a_{m_i}^+ \cdot a_{m_i}^- \quad (17.5)$$

hold. Here no special requirements to elements $b_m^0 \in A^0$, $a_{m_i}^- \in A^-$ and $a_{m_i}^+ \in A^+$ are assumed. Acting by this bilinear operator relation on $|0\rangle$ and using formulas (17.1) and (17.2), we obtain

$$h^0 \phi_m = \epsilon_m \phi_m, \quad \epsilon_m = \epsilon(b_m^0), \quad \phi_m = a_m^+ |0\rangle, \quad m \in Z \quad (17.6)$$

which gives us infinitely many purely algebraic solutions of the spectral problem (17.4). Vectors $\phi_m = a_m^+ |0\rangle$ solving this problem are typical Bethe vectors. Despite the seeming simplicity of this solution, its actual construction is sometimes very complicated. The most non-trivial thing is to find appropriate Bethe vectors for a given hamiltonian. Fortunately, for many physically interesting systems this problem has been successfully solved (see e.g. sections 2 and 8).

18 Modification of an associative product

Up to now the consideration of A -algebras and their representations was rather general: the elements of operator algebra A were considered as *elementary* objects not having any internal structure. In this section we consider a special realization of algebra A whose elements are some *composite operators* and their product is defined in a non-standard way. We demonstrate that such a realization also has a lot of interesting physical applications. In order to distinguish this realization from the general operator algebra A discussed in section 17 we shall use for it the notation \hat{A} and reserve the star-symbol for the product of its elements.

We define the internal structure of the elements $\hat{a} \in \hat{A}$ as follows. We consider them as functions on Z^r whose particular values $a[n]$ are some linear operators on V . The product of two elements \hat{a} and \hat{b} will be denoted by $\hat{a} * \hat{b}$ and defined as

$$(\hat{a} * \hat{b})[n] = a[n + g(\hat{b})] \cdot b[n] \quad (18.1)$$

where the dot means the standard operator multiplication. The action of an element \hat{a} on the vector $v \in V$ is defined as

$$\hat{a}v = a[g(v)]v. \quad (18.2)$$

It is not difficult to check that these definitions do not contradict the axioms 1 – 4.

As in general case, we can introduce the subsets \hat{A}^0 , \hat{A}^+ and \hat{A}^- of algebra \hat{A} consisting of the elements with zero, non-negative and quasi-negative gradings. The values of these subsets on Z^r we respectively denote by $A^0[n]$, $A^+[n]$ and $A^-[n]$.

In order to construct the lowest weight representation V^+ of algebra \hat{A} we need the lowest weight vector $|0\rangle \in V$ satisfying the hat-analogues of equations (17.1) and (17.2):

$$\hat{a}^- |0\rangle = 0, \quad \text{for any } \hat{a}^- \in \hat{A}^-, \quad (18.3)$$

$$\hat{a}^0 |0\rangle = \epsilon(\hat{a}^0) |0\rangle, \quad \text{for any } \hat{a}^0 \in \hat{A}^0. \quad (18.4)$$

If such a vector does exist, we can define the representation space as

$$V^+ = \bigcup_{\hat{a}^+ \in \hat{A}^+} \hat{a}^+ |0\rangle. \quad (18.5)$$

Let us now rewrite formulas (18.3), (18.4) and (18.5) in components. Using definition (18.2) and taking into account that the grading of the lowest weight vector is zero, we obtain:

$$a^-[0]|0\rangle = 0, \quad \text{for any } a^-[0] \in A^-[0], \quad (18.6)$$

$$a^0[0]|0\rangle = \epsilon(a^0[0])|0\rangle, \quad \text{for any } a^0[0] \in A^0[0], \quad (18.7)$$

and

$$V^+ = \bigcup_{a^+[0] \in A^+[0]} a^+[0]|0\rangle. \quad (18.8)$$

From the above consideration it follows that the lowest weight vector $|0\rangle$ is not obliged to be annihilated by all “lowering” operators $a^-[n]$ (with $n \neq 0$) and to be an eigenvector of all “neutral” operators $a^0[n]$ (with $n \neq 0$).

Why the \hat{A} -algebras might be interesting to physicists? There are several reasons for this. Let us try to explain one of them. Assume that we managed to realize one and the same abstract Z^r -graded unital associative algebra (abstract in the sense that it is specified only by some relations between its elements without any concretization of their internal structure) in two different ways: 1) as an operator algebra A and 2) as an algebra of composite operators \hat{A} , i.e. in terms of the dot- and star-products, respectively. Then we can say that algebras A and \hat{A} are isomorphic.

Assume now that algebra A contains two operators a and b commuting with each other:

$$a \cdot b = b \cdot a. \quad (18.9)$$

If we consider this relation from the physical point of view, we can imagine that the operators a and b represent two observables which can be measured simultaneously. After this one can try to make some conclusions about their common spectral properties, etc. In other words, the relation (18.9) may be quite meaningful and informative for physicists.

Due to the assumed isomorphism between the A and \hat{A} algebras, an analogous relation can also be written for the corresponding composite operators \hat{a} and \hat{b} . They also should commute with each other:

$$\hat{a} * \hat{b} = \hat{b} * \hat{a}. \quad (18.10)$$

But what can we now say of formula (18.10) as physicists? At first sight, nothing, because the composite operators and their star products do not have themselves any clear physical meaning. Remember, however, that the composite operators have the meaning of the sets of ordinary operators and their star product can be expressed via ordinary operator product. May be, the relation (18.10) will become more meaningful after rewriting it in components? Let us see. Using formula (18.1), we can write:

$$a[n + g(b)] \cdot b[n] = b[n + g(a)] \cdot a[n] \quad (18.11)$$

Now the relation (18.11) contains everything what we need: the operators and operator products. However, we also see that this formula does not describe any longer a commutativity

of any two objects: in general, this is a certain relation between four different objects! Let us assume, however, that both a and b elements have zero grading. In this case the situation drastically changes: the relation (18.11) takes the form

$$a[n] \cdot b[n] = b[n] \cdot a[n] \quad (18.12)$$

and says us that the operators $a[n]$ and $b[n]$ commute in the standard operator sense for any n . So, we see that we obtained even more than expected. Instead of a single pair of commuting operators which we started with, we obtained an infinite set of pairs of such operators!

What does this mean from the physical point of view? There are many examples of A -algebras containing large families of mutually commuting operators. If the grading of these operators is zero, then it is possible to interpret them as integrals of motion (hamiltonians) of some completely integrable quantum systems. Assume now that we managed to construct a \hat{A} -analog of the above A -algebra which is isomorphic to A . Then we can repeat the reasonings given above and, instead of a single family of commuting operators representing a single integrable system, construct an infinite and discrete set of families of commuting operators which can be treated as integrals of motion of infinitely many different integrable quantum systems.

If we actually want to treat the infinite sets of commuting operators appearing in our scheme as hamiltonians of some physical systems, we should discuss the possible ways of solving the corresponding spectral problems. Let us try to do this in the same way as in the case of general A -algebras (see section 17). Let h^0 be a zero-graded element of algebra A considered as a hamiltonian of some integrable quantum system. Assume that spectral problem for this hamiltonian is exactly solvable. This implies the existence of the relations (17.5) in algebra A . Assume also that we have in our disposal an algebra \hat{A} isomorphic to A . Then, using this isomorphism, we can rewrite relations (17.5) in the \hat{A} -algebraic form:

$$\hat{h}^0 * \hat{a}_m^+ = \hat{a}_m^+ * \hat{b}_m^0 + \sum_i \hat{a}_{m_i}^+ * \hat{a}_{m_i}^-. \quad (18.13)$$

Here, as before, m is an arbitrary non-negative element of Z^r and no special requirements to the elements $\hat{b}_m^0 \in \hat{A}^0$, $\hat{a}_{m_i}^- \in \hat{A}^-$ and $\hat{a}_{m_i}^+ \in \hat{A}^+$ are assumed. Acting by (18.13) on the lowest weight vector $|0\rangle$ and using formulas (18.3) and (18.4) we obtain

$$\hat{h}^0 \phi_m = \epsilon(\hat{b}_m^0) \phi_m, \quad \phi_m = \hat{a}_m^+ |0\rangle. \quad (18.14)$$

Formula (18.14) does not have yet any clear physical meaning. In order to make it physically meaningful, one should rewrite it in components. Taking into account that the grading of vector ϕ_m is m and using formula (18.2), we obtain immediately

$$h^0[m] \phi_m = \epsilon_m \phi_m, \quad \epsilon_m = \epsilon(b_m^0[0]), \quad \phi_m = a_m^+[0]|0\rangle, \quad m \in Z^r \quad (18.15)$$

which gives us a purely algebraic solution of the spectral problems for $h^0[m]$, $m \in Z^r$. Now however, this solution is incomplete, because the vectors $\phi_m = a_m^+[0]|0\rangle$ with any given m form only a negligibly small part of the whole space V^+ . This enables one to make a general conclusion that if the mother integrable system associated with the A -algebra is exactly solvable, i.e. admits a complete algebraic solution of the spectral problem, then the daughter integrable systems associated with the \hat{A} -algebra will be only quasi-exactly solvable, i.e. only some finite parts of their spectra can be constructed algebraically.

19 Construction of modified graded associative algebras

In this section we describe a simple scheme which essentially opens a practical way for building the graded algebras of composite operators. We demonstrate that any A -algebra whose elements admit a special semi-differential realization can be reduced to a certain \hat{A} -algebra which is isomorphic to A .

Assume that each element a of a given algebra A (which is obviously an operator) admits the representation

$$a = e^{t \cdot g(a)} (h(a) + h_i(a) \partial_i + h_{ik}(a) \partial_i \partial_k + \dots) \quad (19.1)$$

where $t = \{t_1, \dots, t_r\}$ is a certain variable vector and $\partial_i = \partial/\partial t_i$, $i = 1, \dots, r$ are partial differential operators with respect to the components of t . The expression in the exponent is a scalar product of two r -dimensional vectors t and $g(a)$. The symbols $h(a), h_i(a), h_{ik}(a), \dots$ denote some t -independent linear operators acting in a certain vector space W . A concrete form of these operators depends on the element a . The summation over repeated indices is also assumed. In this case, the elements of A can be viewed as operators in the space $W = V \times T$ where T denotes the space of all analytic functions of t . Assume also that any vector w of the space W admits the representation

$$w = e^{t \cdot g(w)} v(w) \quad (19.2)$$

where $v(w)$ denotes some t -independent vector of V .

Let us now associate with elements a defined by formula (19.1) some operator-valued functions \hat{a} on Z^r whose particular values, $a[n]$, are operators in the space V defined by the following relations

$$a[n] = h(a) + h_i(a) n_i + h_{ik}(a) n_i n_k + \dots \quad (19.3)$$

It is not difficult to check by means of direct computations that the elements \hat{a} form an \hat{A} -algebra with the products defined by formulas (18.1) and (18.2). This means that any algebra A whose elements admit the representation (19.1) and act in the space V whose vectors are representable form (19.2) can be reduced to a certain \hat{A} algebra.

The next important observation which can be immediately made after analysing formulas (19.1), (19.2) and (19.3) is that the map $A \rightarrow \hat{A}$ we constructed is an isomorphism. This means that if there are some special (polynomial) relations between the elements of the operator algebra A then the map $A \rightarrow \hat{A}$ will induce the same relations between the corresponding elements of \hat{A} . The only difference will concern the definition of the product because the ordinary operator product in A should be replaced by a rather specific product in \hat{A} .

20 Quasi-Lie and quasi-Lie_q algebras

It is known that any Lie or Lie_q algebra \mathcal{L} of the rank r can be considered as a partially Z^r -graded algebra. The grading can be introduced in many different ways, but there is a special, the so-called root grading, which seems to be most natural. In this case the gradings of separate elements of a given algebra form a certain finite subset in Z^r which describes the

root system of an algebra in the basis of r simple roots. This fact enables one to endow with grading also the associated universal enveloping algebra $\mathcal{U}(\mathcal{L})$. This, obviously, will be a full Z^r -grading which makes it possible to consider $\mathcal{U}(\mathcal{L})$ as an A -algebra described in section 17.

Now note that each algebra \mathcal{L} can be realized in the form of differential or pseudo-differential operators in several variables x_1, \dots, x_N . A minimal possible number N of these variables is r (i.e. is equal to the rank of algebra \mathcal{L} and to dimension of grading vectors), and the maximal one is not limited. The differential operators representing the elements of algebra \mathcal{L} may depend additionally on certain numerical parameters characterizing the representation of algebra \mathcal{L} and determining the eigenvalues of zero-graded elements of \mathcal{L} . Therefore, the grading of these parameters also should be equal to zero. From this observation it automatically follows that the variables x_1, \dots, x_N should necessarily be graded. More exactly, at least r of these variables should be graded and these gradings should be linearly independent. Otherwise, it would be impossible to construct from them the generators associated with simple roots whose gradings are linearly independent by definition.

Taking into account the last reasonings, let us now try to reduce the generators L^a of algebra \mathcal{L} to a special canonical form by means of an appropriate change of variables. Note that the existence of r variables with linearly independent gradings makes it possible to choose the new ones in such a way that to have r variables $\rho_1, \rho_2, \dots, \rho_r$ with gradings $\{1, 0, \dots, 0\}, \{0, 1, \dots, 0\}, \dots, \{0, 0, \dots, 1\}$ and all the remaining variables with grading zero. The most general form of generators L^a in the new variables reads

$$L^a = \rho_1^{n_1^a} \dots \rho_r^{n_r^a} \left(H^a + \sum_{i=1}^r H_i^a \rho_i \frac{\partial}{\partial \rho_i} + \dots \right) \quad (20.1)$$

where $n^a = (n_1^a, \dots, n_r^a)$ is the gradings of L^a and H^a, H_i^a , etc. are certain differential operators in the remaining variables. The expression (20.1) is, however, rather cumbersome and therefore, it is convenient to make one extra change of variables, namely, $\rho_i = e^{t_i}$, $i = 1, \dots, r$, which brings (20.1) to the desired canonical form

$$L^a = e^{tg(L^a)} (H^a + H_i^a \partial_i + \dots) \quad (20.2)$$

with $t = \{t_1, \dots, t_r\}$. From this expression it immediately follows that each element of the universal enveloping algebra $\mathcal{U}(\mathcal{L})$ is also representable in the form (19.1) and for this reason $\mathcal{U}(\mathcal{L})$ can be reduced to a \hat{A} -algebra isomorphic to A .

Let us now look at the explicit formulas. Let \mathcal{L} be a Lie algebra. Then the commutation relations for its generators read

$$L^a \cdot L^b - L^b \cdot L^a = C_c^{ab} L^c. \quad (20.3)$$

Rewriting the star-analogues of these relations (which read similarly) in components $L^a[n]$ we obtain the relations

$$L^a[n + g(L^b)] \cdot L^b[n] - L^b[n + g(L^a)] \cdot L^a[n] = C_c^{ab} L^c[n] \quad (20.4)$$

which do not form any Lie algebra with respect to the ordinary operator multiplication. We call such an algebra the quasi-Lie algebra and the expression staying in the left-hand side of

(20.4) — the quasi-commutator. Let now \mathcal{L} be a Lie_q algebra. Then the most general form of the relations for its generators is

$$L^a \cdot L^b = R_{cd}^{ab} L^c \cdot L^d. \quad (20.5)$$

The corresponding star-analogues of these relations being rewritten in components will read

$$L^a[n + g(L^b)] \cdot L^b[n] = R_{cd}^{ab} L^c[n + g(L^d)] \cdot L^d[n]. \quad (20.6)$$

These relations also do not form any Lie_q algebra. We call an algebra described by them the quasi- Lie_q algebra.

Consider an example. In section 7 we already considered the case of quasi- $sl(2)$. Consider now its q -deformed version. Let $\mathcal{L} = sl_q(2)$ with four generators a, b, c, d with gradings $0, -1, +1, 0$, respectively. Then the standard defining relations for this algebra read

$$\begin{aligned} ab &= qba, \\ cd &= qdc, \\ ac &= qca, \\ bd &= qdb, \\ ad - da &= qbc - q^{-1}cb, \\ bc &= cb, \end{aligned} \quad (20.7)$$

with an additional requirement

$$\det_q \begin{pmatrix} a & b \\ c & d \end{pmatrix} = ad - qbc = 1. \quad (20.8)$$

Using the simplified notations $a_n = a[n], b_n = b[n], c_n = c[n], d_n = d[n]$, we can write down the quasi-counterparts of relations (20.7) and (20.8) which respectively will read

$$\begin{aligned} a_{n-1}b_n &= qb_n a_n, \\ c_n d_n &= qd_{n+1} c_n, \\ a_{n+1}c_n &= qc_n a_n, \\ b_n d_n &= qd_{n-1} b_n, \\ a_n d_n - d_n a_n &= qb_{n+1} c_n - q^{-1}c_{n-1} b_n, \\ b_{n+1}c_n &= c_{n-1} b_n \end{aligned} \quad (20.9)$$

and

$$a_n d_n - q b_{n+1} c_n = 1. \quad (20.10)$$

We call an algebra defined by these relations the “quasi- $sl_q(2)$ ”.

21 The main theorem

Collecting the results of sections 17 – 20 we can formulate the following theorem.

Theorem. Let \mathcal{L} be a certain Lie or lie_q algebra defined by commutation relations (20.3) or (20.5). Assume that its generators L^a obey some polynomial relations

$$\sum_{j,k} \sum_{a_1, \dots, a_k} C_{a_k a_{k-1} \dots a_2 a_1}^j L^{a_k} L^{a_{k-1}} \dots L^{a_2} L^{a_1} = 0 \quad (21.1)$$

derivable from the defining relations (20.3) or (20.5) of algebra \mathcal{L} . Then the generators $L^a[n]$ of an abstract quasi- \mathcal{L} algebra defined by quasi-commutation relations (20.4) or (20.6) obey the “quasi” analogues of relations (21.1) which read

$$\sum_{j,k} \sum_{a_1, \dots, a_k} C_{a_k a_{k-1} \dots a_2 a_1}^j L^{a_k}[n + g_{k-1}] L^{a_{k-1}}[n + g_{k-2}] \dots L^{a_2}[n + g_1] L^{a_1}[n + g_0] = 0 \quad (21.2)$$

and in which

$$g_0 = 0, \quad g_k = \sum_{i=1}^k g(L^{a_i}). \quad (21.3)$$

Proof. Denote by A the universal enveloping algebra of algebra \mathcal{L} . It obviously can be considered as a Z^r -graded associative algebra with unit. Using generators $L^a[n]$ of a given quasi- \mathcal{L} algebra, we can construct the composite operators \hat{L}^a whose product is defined by formula (18.1). The algebra $\hat{\mathcal{L}}$ of these operators is isomorphic to \mathcal{L} by construction. Let us now construct the universal enveloping algebra \hat{A} for \mathcal{L} . The isomorphism of algebras \mathcal{L} and $\hat{\mathcal{L}}$ implies the isomorphism of their universal enveloping algebras A and \hat{A} . This, in turn, implies the existence of hat-versions of relations (21.1) in \hat{A} . Rewriting these hat-relations in the components we obtain formulas (21.2) and (21.3), which completes the proof.

Corollary 1. All relations in algebra \mathcal{T}_R imply the existence of analogous quasi-relations in the corresponding quasi- \mathcal{T}_R algebra. These quasi-relations can be obtained in the following simple way. One should consider separately each monomial in an initial \mathcal{T}_R -algebraic relation and endow all its co-factors by one and the same index $n \in Z^r$. After this one should shift the index of each cofactor by a total grading of all right-standing co-factors. Collecting all monomials transformed in such a way together, we obtain the desired form of quasi- \mathcal{T} -algebraic relation.

Corollary 2. Each integrable and exactly solvable model associated with algebra \mathcal{T}_R has an integrable and quasi-exactly solvable counterpart associated with the corresponding quasi- \mathcal{T}_R algebra. Applying the rules given in corollary 1 to formulas describing the form of commuting integrals of motion and determining the structure of Bethe ansatz solutions of integrable and exactly solvable models, we can obtain the corresponding formulas for integrable and quasi-exactly solvable models. In particular, all formulas of sections 3 – 7 and 9 – 12 can be obtained elementary from the analogous formulas of sections 2 and 8.

From these two corollaries it immediately follows that the only non-triviality in constructing the quasi-exactly solvable modifications of integrable exactly solvable models lies in finding the realizations of quasi- \mathcal{T}_R algebras defined by quasi-commutation relations

$$R_{\alpha\beta\gamma\delta}(\lambda - \mu) T_{\gamma\rho}(\lambda, n + g_{\delta\sigma}) T_{\delta\sigma}(\mu, n) = T_{\beta\delta}(\mu, n + g_{\alpha\gamma}) T_{\alpha\gamma}(\lambda, n) R_{\gamma\delta\rho\sigma}(\lambda - \mu) \quad (21.4)$$

with $g_{\alpha\beta} = g(T_{\alpha\beta}(\lambda, n))$. One of the possible ways of constructing such realizations was described in section 19. It is quite obvious however that the proposed realizations do not exhaust the set of all possible ones. This can be seen even from the defining relations (3.1) and (9.1) for quasi- $\mathcal{G}[sl(2)]$ and quasi- $\mathcal{Y}[sl_q(2)]$ algebras which leave a considerable freedom in choosing the form of their generators (see formulas (3.2) and (9.2)). The problem of listing all non-equivalent realizations of these algebras is only a part of the following general one:

Problem. Construct the representation theory for quasi- \mathcal{T}_R algebras.

We expect that solution of this problem may lead to revealing new interesting mathematical structures and notions which necessarily find their applications in the theory of quasi-exactly solvable systems.

22 The classical limit of quasi-Yang-Baxter algebras

It is known that the Gaudin algebra $\mathcal{G}[sl(2)]$ has a well defined classical counterpart — the so-called classical Gaudin algebra. The corresponding Lie-Poisson commutation relations for its generators (which in this case take the meaning of functions on the phase space) read

$$\begin{aligned} \{S^0(\lambda), S^0(\mu)\} &= 0, \\ \{S^+(\lambda), S^+(\mu)\} &= 0, \\ \{S^-(\lambda), S^-(\mu)\} &= 0, \\ \{S^0(\lambda), S^+(\mu)\} &= -(\lambda - \mu)^{-1} \{S^+(\lambda) - S^+(\mu)\}, \\ \{S^0(\lambda), S^-(\mu)\} &= +(\lambda - \mu)^{-1} \{S^-(\lambda) - S^-(\mu)\}, \\ \{S^-(\lambda), S^+(\mu)\} &= -2(\lambda - \mu)^{-1} \{S^0(\lambda) - S^0(\mu)\} \end{aligned} \quad (22.1)$$

and imply the commutativity of functions

$$H(\lambda) = S^0(\lambda)S^0(\lambda) - S^+(\lambda)S^-(\lambda) \quad (22.2)$$

playing the role of integrals of motion of a classical completely integrable system. The latter is usually referred to as the classical Gaudin model.

It is naturally to ask ourselves if there exists any classical limit of the quasi-Gaudin algebra defined in section 3. The answer to this question is positive. In order to demonstrate this we note that the quasi-Gaudin algebra defined by quasi-commutation relations (3.1) admits a class of equivalent representations which can be obtained by an appropriate rescaling the integer parameter n . Indeed, it is not difficult to see that if one uses instead of (3.1) the following modified relations:

$$\begin{aligned} S_{n\hbar}^0(\lambda)S_{n\hbar}^0(\mu) - S_{n\hbar}^0(\mu)S_{n\hbar}^0(\lambda) &= 0, \\ S_{n\hbar+\hbar}^+(\lambda)S_{n\hbar}^+(\mu) - S_{n\hbar+\hbar}^+(\mu)S_{n\hbar}^+(\lambda) &= 0, \\ S_{n\hbar-\hbar}^-(\lambda)S_{n\hbar}^-(\mu) - S_{n\hbar-\hbar}^-(\mu)S_{n\hbar}^-(\lambda) &= 0, \\ S_{n\hbar+\hbar}^0(\lambda)S_{n\hbar}^+(\mu) - S_{n\hbar}^+(\mu)S_{n\hbar}^0(\lambda) &= -\hbar(\lambda - \mu)^{-1} \{S_{n\hbar}^+(\lambda) - S_{n\hbar}^+(\mu)\}, \\ S_{n\hbar-\hbar}^0(\lambda)S_{n\hbar}^-(\mu) - S_{n\hbar}^-(\mu)S_{n\hbar}^0(\lambda) &= +\hbar(\lambda - \mu)^{-1} \{S_{n\hbar}^-(\lambda) - S_{n\hbar}^-(\mu)\}, \\ S_{n\hbar+\hbar}^-(\lambda)S_{n\hbar}^+(\mu) - S_{n\hbar-\hbar}^+(\mu)S_{n\hbar}^-(\lambda) &= -2\hbar(\lambda - \mu)^{-1} \{S_{n\hbar}^0(\lambda) - S_{n\hbar}^0(\mu)\}, \end{aligned} \quad (22.3)$$

in which \hbar is a parameter, then the operators

$$H_{n\hbar}(\lambda) = S_{n\hbar}^0(\lambda)S_{n\hbar}^0(\lambda) - \frac{1}{2}S_{n\hbar+\hbar}^-(\lambda)S_{n\hbar}^+(\lambda) - \frac{1}{2}S_{n\hbar-\hbar}^+(\lambda)S_{n\hbar}^-(\lambda) \quad (22.4)$$

will as before commute with each other for any given n . Let us now rewrite formula (22.3) in the form

$$\begin{aligned} \frac{[S_{n\hbar}^0(\lambda), S_{n\hbar}^0(\mu)]}{\hbar} &= 0, \\ \frac{[S_{n\hbar}^+(\lambda), S_{n\hbar}^+(\mu)]}{\hbar} + \frac{S_{n\hbar+\hbar}^+(\lambda) - S_{n\hbar}^+(\lambda)}{\hbar}S_{n\hbar}^+(\mu) - \frac{S_{n\hbar+\hbar}^+(\mu) - S_{n\hbar}^+(\mu)}{\hbar}S_{n\hbar}^+(\lambda) &= 0, \\ \frac{[S_{n\hbar}^-(\lambda), S_{n\hbar}^-(\mu)]}{\hbar} + \frac{S_{n\hbar-\hbar}^-(\lambda) - S_{n\hbar}^-(\lambda)}{\hbar}S_{n\hbar}^-(\mu) - \frac{S_{n\hbar-\hbar}^-(\mu) - S_{n\hbar}^-(\mu)}{\hbar}S_{n\hbar}^-(\lambda) &= 0, \\ \frac{[S_{n\hbar}^0(\lambda), S_{n\hbar}^+(\mu)]}{\hbar} + \frac{S_{n\hbar+\hbar}^0(\lambda) - S_{n\hbar}^0(\lambda)}{\hbar}S_{n\hbar}^+(\mu) + \frac{S_{n\hbar}^+(\lambda) - S_{n\hbar}^+(\mu)}{\lambda - \mu} &= 0, \\ \frac{[S_{n\hbar}^0(\lambda), S_{n\hbar}^-(\mu)]}{\hbar} + \frac{S_{n\hbar-\hbar}^0(\lambda) - S_{n\hbar}^0(\lambda)}{\hbar}S_{n\hbar}^-(\mu) - \frac{S_{n\hbar}^-(\lambda) - S_{n\hbar}^-(\mu)}{\lambda - \mu} &= 0, \\ \frac{[S_{n\hbar}^-(\lambda), S_{n\hbar}^+(\mu)]}{\hbar} + \frac{S_{n\hbar+\hbar}^-(\lambda) - S_{n\hbar}^-(\lambda)}{\hbar}S_{n\hbar}^+(\mu) - \frac{S_{n\hbar-\hbar}^-(\mu) - S_{n\hbar}^-(\mu)}{\hbar}S_{n\hbar}^-(\lambda) &= \\ &= -2\frac{S_{n\hbar}^0(\lambda) - S_{n\hbar}^0(\mu)}{\lambda - \mu} \end{aligned} \quad (22.5)$$

and apply to it a standard dequantization procedure. For this one should assume that the parameter \hbar is small, n is large, and their product $t = n\hbar$ is finite. Considering t as a new continuous variable and taking the limits $\hbar \rightarrow 0$ and $n \rightarrow \infty$, we can replace the commutators in (22.5) by Poisson brackets and the finite differences by derivatives. The result will read

$$\begin{aligned} \{S_t^0(\lambda), S_t^0(\mu)\} &= 0, \\ \{S_t^+(\lambda), S_t^+(\mu)\} + \dot{S}_t^+(\lambda)S_t^+(\mu) - \dot{S}_t^+(\mu)S_t^+(\lambda) &= 0, \\ \{S_t^-(\lambda), S_t^-(\mu)\} - \dot{S}_t^-(\lambda)S_t^-(\mu) - \dot{S}_t^-(\mu)S_t^-(\lambda) &= 0, \\ \{S_t^0(\lambda), S_t^+(\mu)\} + \dot{S}_t^0(\lambda)S_t^+(\mu) + \frac{S_t^+(\lambda) - S_t^+(\mu)}{\lambda - \mu} &= 0, \\ \{S_t^0(\lambda), S_t^-(\mu)\} + \dot{S}_t^0(\lambda)S_t^-(\mu) - \frac{S_t^-(\lambda) - S_t^-(\mu)}{\lambda - \mu} &= 0, \\ \{S_t^-(\lambda), S_t^+(\mu)\} + \dot{S}_t^-(\lambda)S_t^+(\mu) - \dot{S}_t^+(\mu)S_t^-(\lambda) + 2\frac{S_t^0(\lambda) - S_t^0(\mu)}{\lambda - \mu} &= 0, \end{aligned} \quad (22.6)$$

where the dot denotes the derivative with respect to t . We call an algebra defined by these relations the classical quasi-Gaudin algebra. The existence of this algebra can easily be proved if one replaces n by $n\hbar$ in formulas (5.2), takes the limits $\hbar \rightarrow 0$ and $n \rightarrow \infty$, and checks that the resulting generators (having the same form as in (5.2) but with n replaced by t) obey the relations (22.6). Taking the same limits in formula (22.4), we obtain

$$H_t(\lambda) = S_t^0(\lambda)S_t^0(\lambda) - S_t^+(\lambda)S_t^-(\lambda). \quad (22.7)$$

It is not difficult to see that functions (22.7) are in involution for any given t ,

$$\{H_t(\lambda), H_t(\mu)\} = 0, \quad (22.8)$$

but this is generally not true if the values of t differ from each other. This means that, exactly as in the quantum case, the classical quasi-Gaudin algebra generates an infinite number (or, more exactly, a continuous family) of integrable classical systems. It is natural to call them the classical quasi-Gaudin models.

According to the famous Liouville theorem, all these models are integrable in quadratures and, in this sense, can be qualified as exactly solvable. Remember, however, that the practical usefulness of the Liouville theorem is very low if one deals with classical systems with many degrees of freedom. In general, we simply cannot present their explicit solutions!¹⁸ At present time there is only one constructive way for solving classically integrable systems. It is given by the famous classical inverse scattering method (CISM) (see e.g. [11]) which presents more or less explicit algorithm of constructing action-angle variables for models associated with classical Yang-Baxter algebras¹⁹. We expect that the classical quasi-Gaudin models which we obtained are in fact quasi-exactly solvable in the sense of CISM. This means that using the standard prescriptions of this method and applying them to models (22.7) we probably will obtain only some part of action-angle variables, but not a complete set of them. It would be very desirable to try to verify this conjecture because if it is true then one obtains obtain a remarkable possibility of introducing a concept of quasi-exact solvability in classical mechanics.

Note that the relations (22.6) can be rewritten in more standard form if one introduces a new bracket: $\{\{, \}\}$. Defining it as

$$\{\{a_t, b_t\}\} = \{a_t, b_t\} + g(a_t)\dot{a}_t b_t - g(b_t)\dot{b}_t a_t, \quad (22.9)$$

where $g(a)$, as usually, denotes a grading, we obtain for (22.6):

$$\begin{aligned} \{\{S^0(\lambda), S^0(\mu)\}\} &= 0, \\ \{\{S^+(\lambda), S^+(\mu)\}\} &= 0, \\ \{\{S^-(\lambda), S^-(\mu)\}\} &= 0, \\ \{\{S^0(\lambda), S^+(\mu)\}\} &= -(\lambda - \mu)^{-1}\{S^+(\lambda) - S^+(\mu)\}, \\ \{\{S^0(\lambda), S^-(\mu)\}\} &= +(\lambda - \mu)^{-1}\{S^-(\lambda) - S^-(\mu)\}, \\ \{\{S^-(\lambda), S^+(\mu)\}\} &= -2(\lambda - \mu)^{-1}\{S^0(\lambda) - S^0(\mu)\}. \end{aligned} \quad (22.10)$$

We see that the classical quasi-Gaudin algebra can be considered as an ordinary classical Gaudin algebra realized with respect to the non-standard bracket (22.9). We arrived at the same situation which we had in section 18 for quasi-commutators. This similarity is not

¹⁸This situation is exactly the same as in quantum mechanics: we become happy when a certain quantum problem is reduced to a purely algebraic one and call such a problem exactly solvable (or quasi-exactly solvable). But it does not mean that we are able to solve any algebraic problem explicitly, especially if the number of unknowns is large.

¹⁹The fact that CISM works for integrable models associated with classical Yang-Baxter algebras does not mean that it works for any integrable model.

accidental because the quasi-Poisson bracket (22.9) is nothing else than the classical limit of a quasi-commutator which can be defined as

$$[[a_n, b_n]] = a_{n+g(b)}b_n - b_{n+g(a)}a_n. \quad (22.11)$$

Indeed, following the standard prescriptions of dequantization procedure, we should rescale the relation (22.11) as

$$[[a_{n\hbar}, b_{n\hbar}]] = a_{n\hbar+g(b)\hbar}b_{n\hbar} - b_{n\hbar+g(a)\hbar}a_{n\hbar} \quad (22.12)$$

and then rewrite it in the form

$$[[a_{n\hbar}, b_{n\hbar}]] = [a_{n\hbar}, b_{n\hbar}] + \hbar \frac{a_{n\hbar+g(b)\hbar} - a_{n\hbar}}{\hbar} b_{n\hbar} - \hbar \frac{b_{n\hbar+g(a)\hbar} - b_{n\hbar}}{\hbar} a_{n\hbar} \quad (22.13)$$

Taking now the limits $\hbar \rightarrow 0$ and $n \rightarrow \infty$, introducing a new finite and continuous variable $t = n\hbar$ and using definition (22.9), we obtain

$$[[a_{n\hbar}, b_{n\hbar}]] \rightarrow \hbar \{ \{a_t, b_t\} \}. \quad (22.14)$$

From this analysis it follows that the definition (22.9) works not only for the objects with gradings 0 and ± 1 but for arbitrarily graded ones. One can check that the quasi-Poisson bracket is anti-symmetric

$$\{ \{a_t, b_t\} \} + \{ \{b_t, a_t\} \} = 0 \quad (22.15)$$

and obeys the standard Jacoby identity:

$$\{ \{ \{a_t, b_t\} \}, c_t \} + \{ \{ \{b_t, c_t\} \}, a_t \} + \{ \{ \{c_t, a_t\} \}, b_t \} = 0. \quad (22.16)$$

For zero-graded objects it coincides with an ordinary Poisson bracket

$$\{ \{a_t, b_t\} \} = \{b_t, a_t\}, \quad \text{if } g(a) = g(b) = 0 \quad (22.17)$$

and therefore the commutativity of functions $H_t(\lambda)$ can be considered as an elementary consequence of the commutativity of their Gaudin analogues $H(\lambda)$.

23 The perspectives

We completed the exposition of the proposed R -matrix approach to the problem of quasi-exact solvability. Summarizing, we can say that we found a simple modification of the celebrated quantum inverse scattering method which leads to new wide classes of completely integrable models. The latter are distinguished by the fact they admit an algebraic Bethe ansatz solution only for some limited parts of the spectrum, but not for the whole spectrum, as in the standard version of QISM. Therefore they are typical quasi-exactly solvable models discussed in refs. [41, 43, 45]. An underlying algebra responsible for both the phenomena of complete integrability and quasi-exact solvability was constructed. We called it the “quasi- \mathcal{T}_R algebra” and demonstrated that it can be considered as a very simple (but non-trivial) deformation of the ordinary \mathcal{T}_R algebra of monodromy matrices generated by various solutions of Yang–Baxter equation. This enabled one to claim that we actually have found a new class

of fundamental symmetries of integrable quantum systems. The main attention in the paper was devoted to construction of such deformations for simplest \mathcal{T}_R algebras, i.e. for Gaudin algebra $\mathcal{G}[sl(2)]$ and its quantized version — the Yangian $\mathcal{Y}[sl(2)]$. We also constructed and solved the quasi-exactly solvable models of magnetic chains generated by these algebras in the framework of QISM. The classical versions of quasi- \mathcal{T}_R algebras were discussed briefly and we intend to return to this interesting question in the nearest future.

The main result of this paper lies however in the proof of the fact that any graded associative algebra with unit admits such “quasi” deformation. We showed that this deformation modifies the structure of the elements of an algebra and changes the definition of their associative product. At the same time, despite the numerous attempts we did not manage to find any appropriate deformation of a co-product which would preserve a possible Hopf algebraic structure of the initial algebra. This enables one to make a conjecture that quasi- \mathcal{T}_R algebras which we constructed in this paper *are not* the Hopf algebras²⁰.

The method we proposed in this paper is interesting primarily because it opens a practical way of constructing quasi-exactly solvable deformations of *all* known exactly solvable and completely integrable quantum models obtainable in the framework of standard version of QISM. The list of such models includes:

1) Exactly solvable models of ordinary quantum mechanics. In one-dimensional case these are: the simple harmonic oscillator, the Kratzer and Morse potentials, Pöschel–Teller potential wells, etc. In multi-dimensional case these are: the multi-dimensional harmonic oscillator and also various separable models defined on non-trivial curved manifolds and constructed in ref. [46]. As was demonstrated in [46], all these models can be considered as various degenerate cases of Gaudin models and hence can be obtained in the framework of classical R -matrix approach. This, in turn, means that we can apply to these models our deformation procedure and obtain their quasi-exactly solvable counterparts. This would be a perfect test for the proposed method because the explicit form of all these quasi-exactly solvable models is already known (see e.g. refs. [40, 37, 43, 46]).

2) Exactly solvable models of multi-particle quantum mechanics. These are: the Calogero model, the Calogero-Moser model and other similar models discussed for example in ref. [24]. Recently it was demonstrated that these models can be obtained in the framework of the so-called method of dynamical R -matrices. A main distinguished feature of this method lies in the fact that the R -matrices used in it depend themselves on dynamical variables, i.e. on the components of a monodromy matrix. For us however it does not matter what kind of R -matrices is used in commutation relations of algebra \mathcal{T}_R because of the universality of the proposed deformation method. This means that we can use this method for constructing quasi-exactly solvable versions of multi-particle Calogero and Calogero-Moser models. Note that particular classes of such models have already been constructed in ref. [44] in different way (see also the book [46]).

3) Exactly solvable models of spin chains. These include, first of all, the local chains: the simple 1/2-spin chains with XXX , XXY and XYZ symmetries as well as their higher spin generalizations constructed in refs. [35, 34], and also the various inhomogeneous spin chains with the same symmetries. These models are associated respectively with rational,

²⁰It would be nice to call such algebras “quasi-Hopf algebras”, but, unfortunately, this term is already used in a different sense [7].

trigonometric and elliptic solutions of Yang–Baxter equation. In this paper we discussed only a very particular case of XXX models. Moreover, for these models we presented only the sets of commuting integrals of motion but did not discuss the methods of extracting from these sets the hamiltonians of physically interesting systems.

4) Exactly solvable models of field theory. We know a lot of models of such a sort including non-relativistic models like non-linear Schrödinger equation, sine-Gordon model, various lattice models, and also the relativistic models describing various forms of four-fermion interaction. Using our deformation technique, one can try to construct their quasi-exactly solvable counterparts. This, may be, requires some comments. The quasi-exactly solvable problems which were considered up to now were defined as problems having a certain finite parts of the spectrum constructable in an algebraic way. If the number of such algebraically constructable states is N , then, in order to find them, one should solve a certain N -th order algebraic equation. A fruitfulness of such a concept in quantum mechanics is very well known. However, if we want to have something like field-theoretical quasi-exactly solvable models (or, simply, quasi-exactly solvable models with many degrees of freedom) one should allow N to tend to infinity, because the only physically interesting situations in such models may appear when the number of excitations is infinitely (or, simply, very) large. But for this we should be able to solve the corresponding “algebraic” equations! If we speak only of the algebraizability itself, as, for example, in the partial algebraization method, then we are unable to solve this problem, since the complexity of corresponding algebraic problems catastrophically increases with the increase of N . However, if we have some additional structure in our systems, which is, say, the integrability property, then, as we have seen above, the equations for solving our algebraic problem take the form of Bethe ansatz equations which in the large N limit are well defined and have been studied in detail in many occasions.

5) Exactly solvable models of statistical mechanics. We know that the mathematical structure lying behind such models is again the \mathcal{T}_R algebra, which however has now the meaning of an algebra of transfer matrices. It would be very interesting to try to understand what kind of statistical systems correspond to quasi- \mathcal{T}_R algebras, and in which sense they are quasi-exactly solvable.

Of course, the construction of quasi-exactly solvable models with an infinite number of degrees of freedom in the framework of quasi-QISM approach is a long-term program which hardly can be solved in the nearest future. Now we can only express an optimistic hope that its realization is possible. The reason for our optimism is based on two facts: a) that the standard QISM formalism is well adopted to the study of exactly solvable models with many degrees of freedom, and b) that there is no essential difference between QISM and quasi-QISM, as it follows from the results of our paper. Of course, the quasi-QISM does not exactly coincide with the standard QISM and has many specific features which need a very careful study and understanding. In this connection, I would like to mention two important problems which immediately arise if one wants to give a physical interpretation of the results of the present paper.

The hermiticity problem. Any physically realistic model should be described by hermitian hamiltonians. If one looks at quasi-Gaudin and quasi- XXX models (6.1) and (12.1) which we constructed in our paper, then it becomes obvious that their hermiticity properties strongly depend on the choice of a representation in which the corresponding $\mathcal{G}[sl(2)]$ and

$\mathcal{Y}[sl(2)]$ algebras act. One of trivial possibilities of making these hamiltonians hermitian and the corresponding wavefunctions normalizable is to take the unitary representations of these algebras which, due to their incompactness, are all infinite-dimensional. Are there any other ways? The experience accumulated in the study of integrable and quasi-exactly solvable models enables one to answer this question positively. Indeed, there is a constructive method for building such models (and this is just the inverse method of separation of variables discussed in section 15) which allows one to guarantee the hermiticity of the resulting hamiltonians from very beginning, i.e. by construction (for more detail see e.g. ref. [45, 46]). Unfortunately, in order to apply this method to models (6.1) and (12.1) as well as to other models obtainable in the framework of quasi-QISM, one should first separate the variables in them (we mean here the separation of variables in the generalized Sklyanin sense [30]). This immediately creates two important questions: a) Is there any regular way for performing Sklyanin's procedure of separation of variables for models obtainable in the framework of quasi-QISM? b) Is there any direct way of formulating a criterion of hermiticity immediately in terms of quasi- \mathcal{T}_R algebras and their representations? The second question creates, in turn, a problem of developing some kind of representation theory for quasi- \mathcal{T}_R algebras, etc.

The problem of locality. This is no less important problem on whose positive solution will finally depend a progress in the realization of our program. We know that most of physically interesting models of field theory are local. At the same time the models (6.1) and (12.1) look as typically non-local ones. Then the following question immediately arises: Is it possible to extract from the set of integrals of motion of these models some local hamiltonians? And, if not, then may be there exist other realizations of quasi- $\mathcal{G}[sl(2)]$ and quasi- $\mathcal{Y}[sl(2)]$ algebras for which it would be possible?

I hope that the hermiticity and locality problems will finally find their solutions in the framework of quasi-QISM approach. The more that the construction of hermitian and local quasi-exactly solvable models with many degrees of freedom is possible even in the standard version of QISM. I am addressing the interested reader to ref. [47] in which this question is discussed in detail.

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